LF MATTE 4K KONT 2015

1) LØS INITIALVERDIPROBLEMET

$$y''(t)-4y'(t)+3y(t)=\delta(t-5),\quad y(0)=y'(0)=1$$

·) $\mathcal{L}(y(t))(s):=Y(s)$

·) $\mathcal{L}(y'(t))(s)=s\mathcal{L}(y(t))(s)-y(0)=sY(s)-1$

·) $\mathcal{L}(y''(t))(s)=s\mathcal{L}(y'(t))(s)-y'(0)=s\left(s\mathcal{L}(y(t))(s)-y(0)\right)-y'(0)=s^2Y(s)-s-1$

·) $\mathcal{L}(\delta(t-5))(s)=e^{-5s}$

$\rightsquigarrow\quad y''(t)-4y'(t)+3y(t)=\delta(t-5),\quad y(0)=y'(0)=1$

$\Downarrow$ LAPLACETRANSF

$$s^2Y(s)-s-1-4sY(s)+4+3Y(s)=e^{-5s}$$

$$(s^2-4s+3)Y(s)=s-3+e^{-5s}$$

$$Y(s)=\frac{s-3+e^{-5s}}{(s-3)(s-1)}$$

NULLPUNKTER TIL s^2-4s+3

$$s_{1,2}=2\pm\sqrt{4-3}=2\pm1$$

$$\rightsquigarrow (s^2-4s+3)=(s-3)(s-1)$$

$$\rightsquigarrow Y(s)=\frac{1}{s-1}+\frac{e^{-5s}}{(s-3)(s-1)}$$

DELBRØKOPPSP

$$\frac{1}{(s-3)(s-1)}=\frac{A}{s-3}+\frac{B}{s-1}$$

$$\rightsquigarrow 1=(s-1)A+(s-3)B=s(A+B)-(A+3B)$$

$$\rightsquigarrow \begin{array}{l} A+B=0 \\ A+3B=-1 \end{array} \rightsquigarrow A=-B$$

$$\Downarrow$$

$$2B=-1 \rightsquigarrow B=-\frac{1}{2}\quad A=\frac{1}{2}$$

$$Y(s)=\frac{1}{s-1}+\frac{1}{2}\left(\frac{1}{s-3}-\frac{1}{s-1}\right)e^{-5s}$$

·) $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(t)=e^{at}$

$\mathcal{L}^{-1}\left(e^{-as}F(s)\right)(t)=f(t-a)u(t-a)\qquad [f(t)=\mathcal{L}^{-1}(F(s))(t)]$

$\rightsquigarrow$ ·) $\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)=e^{3t}\rightsquigarrow \mathcal{L}^{-1}\left(e^{-5s}\frac{1}{s-3}\right)=e^{3(t-5)}u(t-5)$

·) $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)=e^{t}\rightsquigarrow \mathcal{L}^{-1}\left(e^{-5s}\frac{1}{s-1}\right)=e^{t-5}u(t-5)$

$$Y(s)=\frac{1}{s-1}+\frac{1}{2}\left(\frac{1}{s-3}-\frac{1}{s-1}\right)e^{-5s}$$

$\Downarrow$ LAPLACETRANSF^{-1}

$$y(t)=e^t+\frac{1}{2}\left(e^{3(t-5)}-e^{t-5}\right)u(t-5).$$

ANNEN MULIGHET: $Y(s) = \frac{1}{s-1} + \frac{e^{-5s}}{(s-1)(s-3)} = \frac{1}{s-1} + \frac{e^{-5s}}{s^2-4s+3} = \frac{1}{s-1} + \frac{e^{-5s}}{(s-2)^2-1}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)(t) = \sinh(t)$$

$$\mathcal{L}^{-1}(F(s-a))(t) = e^{at}f(t) \rightsquigarrow \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2-1}\right)(t) = e^{2t}\sinh(t)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-5s}}{(s-2)^2-1}\right)(t) = u(t-5)\,e^{2(t-5)}\sinh(t-5)$$

2) LA FUNKSJONEN f VÆRE DEFINERT VED $f(x)=\cos(x)$ FOR $0<x<\pi$

a) FINN FOURIERSINUSREKKEN TIL $f(x)$

$\cos'(x) = -\sin(x)$
$\sin'(x) = \cos(x)$

$\cdot)$ $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ MED $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)\cos(x)\,dx = \frac{2}{\pi}\sin(nx)\sin(x)\Big|_0^{\pi} - \frac{2}{\pi}n\int_0^{\pi}\cos(nx)\sin(x)\,dx$$

$$= \frac{2}{\pi}n\cos(nx)\cos(x)\Big|_0^{\pi} + \frac{2}{\pi}n^2\int_0^{\pi}\sin(nx)\cos(x)\,dx$$

$$= \frac{2}{\pi}n(-1)^n - \frac{2}{\pi}n\,1 + n^2 b_n$$

$\Rightarrow n \neq 1$: $\quad b_n = \frac{2}{\pi}\frac{n}{1-n^2}\left((-1)^n - 1\right)$

b_1: $\quad b_1 = \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(x)\,dx = \frac{2}{\pi}\sin(x)^2\Big|_0^{\pi} - \frac{2}{\pi}1\int_0^{\pi}\cos(x)\sin(x)\,dx = -b_1$

$\Rightarrow b_1 = 0$

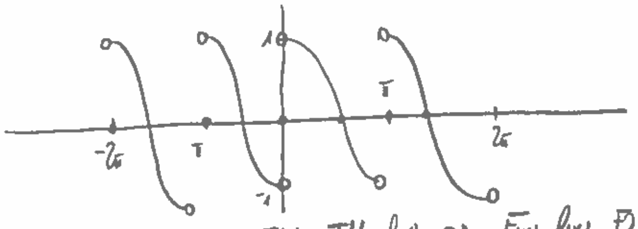
$$\Rightarrow f(x) = \sum_{n=2}^{\infty}\frac{2}{\pi}\frac{n}{1-n^2}\left((-1)^n-1\right)\sin(nx).$$

$$= \sum_{\ell=0}^{\infty}\frac{2}{\pi}\frac{2\ell+1}{1-(2\ell+1)^2}(-2)\sin((2\ell+1)x) = \frac{4}{\pi}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{(2\ell+1)^2-1}\sin((2\ell+1)x)$$

b) SKISSER SUMMEN AV FOURIERSINUSREKKEN TIL $f(x)$ PÅ INTERVALLET $[-2\pi, 2\pi]$.
FINN VERDIEN TIL FOURIERSINUSREKKEN TIL $f(x)$ I PUNKTENE $x=-\frac{\pi}{4}$, $x=0$ OG $x=\frac{\pi}{2}$

FOURIERSINUSREKKEN TIL $f(x) \approx$ ODDE UTVIDELSE TIL $f(x)$.

$\Rightarrow$ FOURIERSINUSREKKEN TIL $f(x)$:



LA $F(x)$... FOURIERSINUSREKKEN TIL $f(x)$ $\rightsquigarrow$ $F(x)=f(x)$ FOR $x=(0,\pi)$. (f KONT PÅ $(0,\pi)$)

$$\Rightarrow F\left(-\frac{\pi}{4}\right) = F\left(\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$F\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0.$$

$F(0) = \frac{1}{2}F(0+) + F(0-) = \frac{1}{2}(F(0+) - F(0+)) = 0.$ (F DISKONT I $x=0$)

3) LA C VÆRE SIRKELEN $\{z\in\mathbb{C} : |z-2|=2\}$ ORIENTERT MOT KLOKKA. FINN VERDIEN AV LINJEINTEGRALET

$$\oint_C \frac{1}{(z-1)(z-7)}\,dz$$

INTEGRAND $\frac{1}{(z-1)(z-7)}$: TO POLER AV ORDNING 1: $z=1$, $z=7$.

$|1-2|=|1|<2 \Rightarrow z=1$ PÅ INNSIDEN

$|7-2|=5>2 \Rightarrow z=7$ PÅ UTSIDEN

$$\Rightarrow \oint_C \frac{1}{(z-1)(z-7)}\,dz = 2\pi i \operatorname*{Res}_{z=1} \frac{1}{(z-1)(z-7)} = 2\pi i\,\frac{1}{1-6} = -\frac{\pi i}{3}$$

4) VIS VED HJELP AV CAUCHY-RIEMANNLIGNINGENE AT $f(z)=ze^{iz}$ ER EN HEL FUNKSJON, DVS AT $f(z)$ ER ANALYTISK I HELE $\mathbb{C}$

BRUK THM 13.4.2.

$$f(z)=ze^{iz}=(x+iy)e^{i(x+iy)}=(x+iy)e^{-y}e^{ix}=e^{-y}(x+iy)(\cos(x)+i\sin(x))$$
$$=(x\cos(x)-y\sin(x))e^{-y}+i(y\cos(x)+x\sin(x))e^{-y}$$

$\Rightarrow u(x,y)=(x\cos(x)-y\sin(x))e^{-y} \rightsquigarrow u_x(x,y)=(\cos(x)-x\sin(x)-y\cos(x))e^{-y}$ → KONT

$u_y(x,y)=(-\sin(x)-x\cos(x)+y\sin(x))e^{-y}$ → KONT

$v(x,y)=(y\cos(x)+x\sin(x))e^{-y} \rightsquigarrow v_x(x,y)=(-y\sin(x)+\sin(x)+x\cos(x))e^{-y}$ → KONT

$v_y(x,y)=(\cos(x)-y\cos(x)-x\sin(x))e^{-y}$ → KONT

$u_x(x,y)=(\cos(x)-x\sin(x)-y\cos(x))e^{-y}=v_y(x,y)$ ✓

$u_y(x,y)=(-\sin(x)-x\cos(x)+y\sin(x))e^{-y}=-v_x(x,y)$ ✓

5) LA $R>0$ OG S_R VÆRE HALVSIRKELEN MED PARAMETRISERING $z(\theta)=Re^{i\theta}$ $0\leq\theta\leq\pi$

LA $x\geq 0$ OG BRUK ML-ULIKHETEN TIL Å VISE AT

$$\lim_{R\to\infty}\int_{S_R}\frac{1}{4+w^4}e^{iwx}\,dw=0.$$

$$\int_{S_R}\frac{1}{4+w^4}e^{+iwx}\,dw=\int_0^{\pi}\frac{1}{4+R^4e^{4i\theta}}e^{+iR(\cos(\theta)+i\sin\theta)}\,iRe^{i\theta}\,d\theta$$

$$\rightarrow \left|\int_{S_R}\frac{1}{4+w^4}e^{+iwx}\,dw\right|\leq\pi\frac{1}{(R^4-4)}R\rightarrow 0 \quad (R\to\infty)$$

BRUKTE $|e^{i\theta}|=1$ OG $-R\sin\theta<0$ OG DERMED $|e^{iR(\cos(\theta)+i\sin\theta)}|\leq e^{-R\sin(\theta)}\leq 1$.

6)a) FINN OG KLASSIFISER DE SINGULÆRE PUNKTENE TIL FUNKSJONEN

$$f(z)=\frac{z^{4n}-1}{z^{2n+1}} \quad n=1,2,\dots$$

FINN LAURENTREKKEN TIL $f(z)$ OM $z=0$ OG REGN UT RESIDYET I $z=0$.

$$f(z)=\frac{z^{4n}-1}{z^{2n+1}}=z^{2n-1}-\frac{1}{z^{2n+1}} \rightarrow$$

SINGULÆRE PUNKTENE

$z=0$: POL AV ORDEN $2n+1$

LAURENTREKKEN TIL $f(z)$: $-\frac{1}{z^{2n+1}}+z^{2n-1}$

b) BRUK RESIDYREGNING FOR Å VISE AT

$$\int_0^{2\pi}\cos(n\theta)\sin(n\theta)\,d\theta=0 \quad n=1,2,\dots$$

BRUK SUBSTITUSJON $z=e^{i\theta}=\cos(\theta)+i\sin(\theta)$

$\Rightarrow e^{-i\theta}=\cos(\theta)-i\sin(\theta)$

$$\Rightarrow \cos(\theta)=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin(\theta)=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})=\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$\Rightarrow \cos(n\theta)=\frac{1}{2}\left(e^{in\theta}+e^{-in\theta}\right)=\frac{1}{2}\left(z^n+\frac{1}{z^n}\right)$$

$$\sin(n\theta)=\frac{1}{2i}\left(e^{in\theta}-e^{-in\theta}\right)=\frac{1}{2i}\left(z^n-\frac{1}{z^n}\right)$$

$$\frac{dz}{d\theta}=ie^{i\theta}=iz$$

$$\Rightarrow \int_0^{2\pi}\cos(n\theta)\sin(n\theta)\,d\theta=\oint_C \frac{1}{2}\left(z^n+\frac{1}{z^n}\right)\frac{1}{2i}\left(z^n-\frac{1}{z^n}\right)\frac{1}{iz}\,dz=\oint_C -\frac{1}{4}\underbrace{\left(z^{2n-1}-\frac{1}{z^{2n+1}}\right)}_{f(z)\text{ FRA A}}dz$$

$C \downarrow$ ENHETSSIRKEL MOT KLOKKA

$$=-2\pi i\,\frac{1}{4}\,b_1$$

MED b_1 = KOEFF TIL $\frac{1}{z}$ I LAURENTREKKEN TIL $f(z)$.

$2n-1\geq 1$ FOR $n=1,2,\dots$

$2n+1\geq 3$ FOR $n=1,2,\dots$

$\Downarrow$

$b_1=0$

$$\Rightarrow \int_0^{2\pi}\cos(n\theta)\sin(n\theta)\,d\theta=0$$

TIL 6a) $\operatorname{Res} f(z)$ = KOEFFIZIENTEN TIL $\frac{1}{z}$ I LAURENTREKKEN

$=0$

10) FINN ALLE LØSNINGER PÅ FORMEN $u(x,t)=F(x)G(t)$ SOM TILFREDSSTILLER DEN PARTIELLE DIFFERENSIALLIGNINGEN

$$u_t(x,t)+(1+t^2)(2u(x,t)-u_{xx}(x,t))=0 \qquad x\in[0,\tfrac{\pi}{2}],\ t>0$$

OG RANDBETINGELSENE

$$u(0,t)=u_x(\tfrac{\pi}{2},t)=0 \quad t\geq 0$$

$$u(x,t)=F(x)G(t) \longrightarrow u(0,t)=F(0)G(t)=0 \quad t\geq 0 \Rightarrow \begin{cases} F(0)=0 \ \checkmark \\ G(t)=0\ \forall t\geq 0 \rightsquigarrow u\equiv 0 \ \text{(UØNSKET)} \end{cases}$$

$$u_x(\tfrac{\pi}{2},t)=F'(\tfrac{\pi}{2})G(t)=0 \quad t\geq 0 \Rightarrow \begin{cases} F'(\tfrac{\pi}{2})=0 \ \checkmark \\ G(t)=0\ \forall t\geq 0 \rightsquigarrow u\equiv 0 \ \text{(UØNSKET)} \end{cases}$$

$\rightsquigarrow$ RANDBETINGELSENE $\Rightarrow F(0)=0=F'(\tfrac{\pi}{2})$

$$u_t(x,t)+(1+t^2)(2u(x,t)-u_{xx}(x,t))=0 \rightsquigarrow F(x)G'(t)+(1+t^2)(2F(x)G(t)-F''(x)G(t))=0$$

$$\frac{G'(t)}{(1+t^2)G(t)}=\frac{2F(x)-F''(x)}{F(x)}=\lambda \ \ldots \text{ KONSTANT}$$

$\cdot)$ $\dfrac{2F(x)-F''(x)}{F(x)}=\lambda \Rightarrow (2-\lambda)F(x)=F''(x)$

$\cdot)$ $(2-\lambda)=0 \Rightarrow F(x)=A+Bx$

$$\left.\begin{array}{l} F(0)=0 \Rightarrow A=0 \\ F'(\tfrac{\pi}{2})=0 \Rightarrow B=0 \end{array}\right\} \Rightarrow u\equiv 0 \ \text{(UØNSKET)}$$

$\cdot)$ $(2-\lambda)>0 \Rightarrow F(x)=Ae^{\sqrt{2-\lambda}\,x}+Be^{-\sqrt{2-\lambda}\,x}$

$$\left.\begin{array}{l} \Rightarrow F(0)=0 \Rightarrow A+B=0 \\ F'(\tfrac{\pi}{2})=0 \Rightarrow A(e^{\sqrt{2-\lambda}\frac{\pi}{2}}-e^{-\sqrt{2-\lambda}\frac{\pi}{2}})=0 \end{array}\right\} \Rightarrow F=0 \Rightarrow u\equiv 0 \ \text{(UØNSKET)}$$

$\cdot)$ $(2-\lambda)<0 \Rightarrow F(x)=A\cos(\sqrt{\lambda-2}\,x)+B\sin(\sqrt{\lambda-2}\,x)$

$F(0)=0 \Rightarrow A=0$

$$F'(\tfrac{\pi}{2})=0 \Rightarrow B\sqrt{\lambda-2}\cos(\sqrt{\lambda-2}\tfrac{\pi}{2})=0 \Rightarrow \begin{cases} B=0 \Rightarrow u\equiv 0 \ \text{(UØNSKET)} \\ B\neq 0 \Rightarrow \sqrt{\lambda-2}\tfrac{\pi}{2}=(2n+1)\tfrac{\pi}{2} \quad n=0,1,2,\ldots \end{cases}$$

$$\sqrt{\lambda-2}=(2n+1) \quad n=0,1,2,\ldots$$

$\Rightarrow$ LET $F_n(x)=B\sin((2n+1)x)$

$\Rightarrow G_n(t)$ OPPFYLLER: $\dfrac{G_n'(t)}{(1+t^2)G_n(t)}=\lambda=(2n+1)^2+2$

$\sqrt{\lambda-2}=(2n+1)$

$\lambda-2=4n^2+4n+1$

$\lambda=4n^2+4n+3$

$\Rightarrow \ln(G_n(t))-\ln(G_n(0))=\lambda(t+\tfrac{t^3}{3})$

$\Rightarrow G_n(t)=G_n(0)e^{\lambda(t+\frac{t^3}{3})}$

$\Rightarrow$ LØSNINGER PÅ FORMEN:

$$u(x,t)=\sum_{n=0}^{\infty}B_n\sin((2n+1)x)e^{((2n+1)^2+2)(t+\frac{t^3}{3})}$$

b) FINN EN LØSNING SOM TILFREDSSTILLER (1) OG (2) OG I TILLEGG RANDBETINGELSEN:

$$u(x,0) = \sin(3x) + \sin(17x) \qquad x \in [0, \tfrac{\pi}{2}]$$

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin((2n+1)x) \quad \Rightarrow \quad \begin{array}{l} B_1 = 1 \\ B_8 = 1 \\ \text{ELLERS } B_i = 0. \end{array}$$

$$\rightarrow u(x,t) = \sin(3x)e^{11(t+\frac{t^3}{3})} + \sin(17x)e^{291(t+\frac{t^3}{3})}$$