$F = S_1, S_2, S_3$ are shown in Fig. 269 in Sec. 11.2, and $F = S_{20}$ is shown in Fig. 279. Although $|f(x) - F(x)|$ is large at $\pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where "waves" remain owing to the "Gibbs phenomenon," which we shall discuss in the next section.

Can you think of functions f for which E^* decreases more quickly with increasing N? ■

PROBLEM SET 11.4

1. CAS Problem. Do the numeric and graphic work in Example 1 in the text.

2–5 MINIMUM SQUARE ERROR

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi < x < \pi$ is minimum. Compute the minimum value for $N = 1, 2, \cdots, 5$ (or also for larger values if you have a CAS).

2. $f(x) = x \quad (-\pi < x < \pi)$

3. $f(x) = |x| \quad (-\pi < x < \pi)$

4. $f(x) = x^2 \quad (-\pi < x < \pi)$

5. $f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases}$

6. Why are the square errors in Prob. 5 substantially larger than in Prob. 3?

7. $f(x) = x^3 \quad (-\pi < x < \pi)$

8. $f(x) = |\sin x| \quad (-\pi < x < \pi)$, full-wave rectifier

9. Monotonicity. Show that the minimum square error (6) is a monotone decreasing function of N. How can you use this in practice?

10. CAS EXPERIMENT. Size and Decrease of E^*. Compare the size of the minimum square error E^* for functions of your choice. Find experimentally the factors on which the decrease of E^* with N depends. For each function considered find the smallest N such that $E^* < 0.1$.

11–15 PARSEVALS'S IDENTITY

Using (8), prove that the series has the indicated sum. Compute the first few partial sums to see that the convergence is rapid.

11. $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8} = 1.233700550$

Use Example 1 in Sec. 11.1.

12. $1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots = \dfrac{\pi^4}{90} = 1.082323234$

Use Prob. 14 in Sec. 11.1.

13. $1 + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \cdots = \dfrac{\pi^4}{96} = 1.014678032$

Use Prob. 17 in Sec. 11.1.

14. $\displaystyle\int_{-\pi}^{\pi} \cos^4 x\, dx = \frac{3\pi}{4}$

15. $\displaystyle\int_{-\pi}^{\pi} \cos^6 x\, dx = \frac{5\pi}{8}$

11.5 Sturm–Liouville Problems. Orthogonal Functions

The idea of the Fourier series was to represent general periodic functions in terms of cosines and sines. The latter formed a *trigonometric system.* This trigonometric system has the desirable property of orthogonality which allows us to compute the coefficient of the Fourier series by the Euler formulas.

The question then arises, can this approach be generalized? That is, can we replace the trigonometric system of Sec. 11.1 by other *orthogonal systems* (*sets of other orthogonal functions*)? The answer is "yes" and will lead to generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series in Sec. 11.6.

To prepare for this generalization, we first have to introduce the concept of a Sturm–Liouville Problem. (The motivation for this approach will become clear as you read on.) Consider a second-order ODE of the form