Exam TMA4120 MATHEMATICS 4K

Monday 19.12.2012, Time: $9^{\underline{00}} - 13^{\underline{00}}$

English
Hjelpemidler (Kode C): Bestemt kalkulator (HP 30S eller Citizen SR-270X), Rottmann: *Matematisk formelsamling*

Problem 1.
a. Determine the value

$$\left|\frac{(1+i)^6}{i^3(1+4i)^2}\right|.$$

Solution
We have

$$\left|\frac{(1+i)^6}{i^3(1+4i)^2}\right| = \frac{|1+i|^6}{|i^3||1+4i|^2}.$$

Further

$$|1+i|^6 = 2^3 = 8; \quad |i|^3 = 1; \quad |1+4i|^2 = 17.$$

Finally

$$\left|\frac{(1+i)^6}{i^3(1+4i)^2}\right| = \underline{\frac{8}{17}}.$$

b. Let $\omega^3 = 1$ and Im $\omega \neq 0$. Find

$$\omega^2 + \omega + 1.$$

Solution 1
$\omega^3 = 1 \;\Rightarrow\; \omega^3 - 1 = 0$. Or $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$. Since $\Im\omega \neq 0$ we have $\omega \neq 1$. Therefore $\underline{\omega^2 + \omega + 1 = 0}$.

Solution 2
Equation $\omega^3 = 1$ has the following solutions: $\omega_0 = 1$, $\omega_1 = e^{2i\pi/3}$, and $\omega_2 = e^{4i\pi/3}$. Since $\Im\omega_0 = 0$ we have to find

$$\omega_1^2 + \omega_1 + 1 \quad \text{and} \quad \omega_2^2 + \omega_2 + 1$$

only. We have

$$\omega_1 = e^{2i\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

and

$$\omega_1^2 = e^{4i\pi/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Respectively

$$\underline{\omega_1^2 + \omega_1 + 1 = 0}.$$

Relation $\omega_2^2+\omega_2+1=0$ can be proved similarly or just mention that $\omega_2=\overline{\omega_1}$, hence $\omega_2^2+\omega_2+1=\overline{\omega_1^2+\omega_1+1}=0$.

Problem 2.

a. Find the Laplace transform $F(s)=\mathcal{L}f(s)$ of the function $f(t)=te^{-t}\cos 2t$.

Solution.

We have

$$\mathcal{L}(\cos 2t)(s)=\frac{2}{s^2+4}.$$

First shifting theorem:

$$\mathcal{L}(e^{-t}\cos 2t)(s)=\frac{s+1}{(s+1)^2+4}.$$

Differentiation of transform formula:

$$\mathcal{L}(te^{-t}\cos 2t)(s)=-\left(\frac{s+1}{(s+1)^2+4}\right)'=\underline{\frac{(s+1)^2-4}{[(s+1)^2+4]^2}}.$$

Comment: There are many other ways to solve this problem. Each is OK so far it is correct.

b. Solve the integral equation

$$y(t)=e^t\left\{1+\int_0^t e^{-\tau}y(\tau)d\tau\right\},\quad t\ge 0.$$

Solution 1

We rewrite the equation as

$$y(t)=e^t+\int_0^t e^{t-\tau}y(\tau)d\tau,\quad t\ge 0, \qquad (*)$$

so the integral in the right-hand side is a convolution of $y(t)$ and e^t.

Denote $Y(s)=\mathcal{L}y(s)$ and use that $\mathcal{L}(e^t)(s)=(s-1)^{-1}$. We then have

$$(*)\ \Rightarrow\ Y(s)=\frac{1}{s-1}+\frac{1}{s-1}Y(s)\ \Rightarrow\ Y(s)=\frac{1}{s-2}\ \Rightarrow\ \underline{y(t)=e^{2t}}.$$

Solution 2

We rewrite the equation as

$$y(t)e^{-t}=1+\int_0^t e^{-\tau}y(\tau)d\tau,\quad t\ge 0,$$

or

$$z(t)=1+\int_0^t z(\tau)d\tau, \qquad (**)$$

where $z(t)=e^{-t}y(t)$. The integral in the right hand side of (**) is the convolution of the Heaviside function and $z(t)$. Let $Z(s)=\mathcal{L}z(s)$. The Laplace transform of (**) gives

$$Z(s)=\frac{1}{s}+\frac{1}{s}Z(s)\ \Rightarrow\ Z(s)=\frac{1}{s-1}\ \Rightarrow\ z(t)=e^t\ \Rightarrow\underline{y(t)=e^{2t}}.$$

Problem 3. Let $f(x)$ be the 2π-periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < 0; \\ 1, & 0 < x < \pi. \end{cases}$$

Find its Fourier series. Then determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Hint: Parseval's formula.

Solution

Fourier series:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}; \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

We have

$$c_0 = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2}; \quad c_n = \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx = \frac{1}{-2in\pi} e^{inx}|_0^{\pi} = \begin{cases} 0, & n \text{ is even}; \\ \frac{1}{in\pi}, & n \text{ is odd}. \end{cases}$$

Finally

$$\underline{f(x) = \frac{1}{2} + \frac{1}{i\pi} \sum_{l=-\infty}^{\infty} \frac{1}{2l-1} e^{i(2l-1)x}}.$$

Parseval formula:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

In our case $1/2\pi \int_{-\pi}^{\pi} |f(x)|^2 dx = 1/2$, therefore

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{\pi^2} \sum_{l=-\infty}^{\infty} \frac{1}{(2l-1)^2} = \frac{1}{4} + \frac{2}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{(2l-1)^2}.$$

Finally

$$\underline{\sum_{l=1}^{\infty} \frac{1}{(2l-1)^2} = \frac{\pi^2}{8}}.$$

Comment. You can also use Parseval's formula for sin and cos Fourier series. It will lead you to the same result of course.

Problem 4. The function $u(x,t)$ satisfies the equation

$$u_{xx} = u_t - u, \quad 0 < x < \pi, \quad t \geq 0 \qquad (\dagger)$$

and the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \geq 0. \qquad (\ddagger)$$

a. Find all solutions of this problem having the form $u(x,t) = X(x)T(t)$.

Solution

Let $u(x,t) = X(x)T(t)$. Then (†) $\Rightarrow X''(x)T(t) = X(x)T'(t) + X(x)T(t) \Rightarrow$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} - 1 = k,$$

here k is an unknown constant.

Taking (‡) into account we have

$$X''(x) - kX(x) = 0,\ 0 < x < \pi, \quad X(0) = 0,\ X(\pi) = 0.$$

and

$$T'(t) - (k+1)T(t) = 0,\ t > 0.$$

The standard analysis shows that non-trivial solutions exist for $k = -n^2$, $n = 1, 2, \ldots$ and the corresponding functions X_n and T_n are $X_n(x) = \sin nx$ and $T_n(t) = e^{(1-n^2)t}$, so

$$\underline{u_n(x,t) = b_n e^{(1-n^2)t} \sin nx.}$$

b. Find the solution $u(x,t)$ which also satisfies the initial condition

$$u(x,0) = \sin^2 x, \quad 0 < x < \pi.$$

Solution

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{(1-n^2)t} \sin nx, \qquad (\bullet)$$

where b_n are the coefficients in the expansion

$$\sin^2 x = \sum_{n=1}^{\infty} b_n \sin nx, \quad 0 < x < \pi.$$

Finding b_n:

$$b_n = \frac{2}{\pi}\int_0^\pi \sin^2 x \sin nx dx = \frac{2}{\pi}\int_0^\pi \frac{1-\cos 2x}{2} \sin nx dx =$$
$$\underbrace{\frac{1}{\pi}\int_o^\pi \sin nx dx}_{J_{1,n}} - \underbrace{\frac{1}{\pi}\int_0^\pi \cos 2x \sin nx dx}_{J_{2,n}}.$$

We have

$$J_{1,n} = \frac{1}{\pi}\int_o^\pi \sin nx dx = -\frac{1}{n\pi}\cos nx|_0^\pi = \begin{cases} \frac{2}{n\pi}, & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases}$$

In order to find $J_{2,n}$ we use the relation $\sin\alpha\cos\beta = (\sin(\alpha+\beta) + \sin(\alpha-\beta))/2$. Then

$$J_{2,n} = \frac{1}{\pi}\int_0^\pi \cos 2x \sin nx dx = \frac{1}{2\pi}\int_0^\pi \sin(n+2)x dx + \frac{1}{2\pi}\int_0^\pi \sin(n-2)x dx.$$

Let $n \neq 2$ then (similarly to calculation of $J_{1,n}$):

$$J_{2,n} = \begin{cases} \frac{1}{(n+2)\pi} + \frac{1}{(n-2)\pi}, & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases}$$

Direct calculation: $b_2 = 0$.

Finally

$$b_n = \begin{cases} \frac{1}{\pi(n-2)} + \frac{2}{\pi n} + \frac{1}{(n+2)\pi}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

Comment You may substitute this expression into (•), but there is no need for it. Minor punishment will be applied for those who made mistake in calculating the integrals.

Problem 5. Find the Fourier transform

$$\hat{f}(w) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x) e^{-iwx} dx,$$

where

$$f(x) = \begin{cases} 1, & \text{if } -1 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Use the formula for the inverse Fourier transform in order to find the value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin(2w)\cos w}{w} dw.$$

Solution

Fourier transform:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx = \underline{\sqrt{\frac{2}{\pi}} \frac{\sin w}{w}}.$$

Inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin u}{u} e^{iux} du = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} \cos ux \, du + i \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} \sin ux \, du.$$

The last term in the right-hand side vanishes because the integrand is odd, so we have

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} \cos ux \, du.$$

Now change variables $u = 2w$:

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2w}{w} \cos 2wx \, dw,$$

and set $x = 1/2$ and multiply the both sides by π. We obtain

$$\underline{\pi = \pi f\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} \frac{\sin 2w}{w} \cos w \, dw.}$$

Problem 6.

a. Find the singular points of the function

$$f(z) = \frac{e^{5iz}}{z^2 - 2z + 2},$$

classify them (poles, essential singularities, removable singularities), and determine the residues.

Solutions

The singular points of f are zeroes of the denominator i.e. the points $z_1 = 1 + i$ and $z_2 = 1 - i$. They are simple poles. Therefore the residues can be defined by the formula

$$\text{Res}_{z_k} f = \frac{e^{5iz}}{(z^2 - 2z + 2)'}|_{z=z_k}, \quad k = 1, 2.$$

In particular

$$\underline{\text{Res}_{1+i} f = \frac{1}{2i} e^{5(-1+i)}}, \quad \underline{\text{Res}_{1-i} \frac{-1}{2i} e^{5(1+i)}}.$$

b. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin 5x}{x^2 - 2x + 2} dx.$$

Explain your procedure.

Solution

We have

$$\int_{-\infty}^{\infty} \frac{\sin 5x}{x^2 - 2x + 2} dx = \underbrace{\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{5ix}}{x^2 - 2x + 2} dx}_{I_1} - \underbrace{\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{-5ix}}{x^2 - 2x + 2} dx}_{I_2}.$$

The function e^{5iz} decays in the upper half-plane so in order to complement the integral I_1 over the segments of real line to a closed curve one has to use the half-circles located in the upper half-plane. Therefore

$$I_1 = \pi \text{Res}_{1+i} \left(\frac{e^{5iz}}{z^2 - 2z + 2} \right) = \frac{\pi}{2i} e^{5(-1+i)}.$$

The function e^{-5iz} decays in the lower half-plane so in order to complement the integral I_2 over the segments of real line to a closed curve one has to use the half-circles located in the lower half-plane. Also orientation should be taken into account. We obtain

$$I_2 = -\pi \text{Res}_{1-i} \left(\frac{e^{-5iz}}{z^2 - 2z + 2} \right) = \frac{\pi}{2i} e^{5(-1-i)}.$$

Finally

$$\int_{-\infty}^{\infty} \frac{\sin 5x}{x^2 - 2x + 2} dx = I_1 - I_2 = \underline{\pi e^{-5} \sin 5}.$$

Another, and actually simpler way of finding I_2 is to mention that $2iI_1 = -2i\bar{I}_2$. Then you can avoid extra calculations.