TMA 4120 MAT. 4K 14.XII.2009 Peter Lindqvist

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① $e^{2z} = i$, $z = x + iy = ?$

$$e^{2x} e^{+2iy} = e^{i \cdot \frac{\pi}{2}} \iff \begin{cases} e^{2x} = 1 \\ 2y = \frac{\pi}{2} + 2n\pi \end{cases}$$

$$\iff x = 0 \ \& \ y = \frac{\pi}{4} + n\pi$$

$$z = i\left(\frac{\pi}{4} + n\pi\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

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② $\begin{cases} y''(t) + 100\,y(t) = \delta(t-2) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$

$$\mathcal{L}\{\delta(t-2)\} = \text{"}\int_0^\infty e^{-st}\delta(t-2)\,dt\text{"} = e^{-2s}$$

$$s^2 Y(s) + 100\,Y(s) = e^{-2s}$$

$$Y(s) = \frac{e^{-2s}}{s^2+100} = \frac{1}{10}\underbrace{\frac{10}{s^2+10^2}}_{\frac{1}{10}\mathcal{L}\{\sin(10t)\}} e^{-2s}$$

$$y(t) = \frac{1}{10}\sin(10(t-2)) \cdot u(t-2)$$

(3a) $u_t = u_{xx} \quad (0 < x < \pi, \ t > 0)$

$u_x(0,t) = 0 = u_x(\pi, t), \quad t > 0$

Neumann

$u(x,t) = X(x)T(t), \quad u_x(x,t) = X'(x)T(t)$

$$X''T = XT'$$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda T = 0 \iff \end{cases}$$

$T = Ce^{-\lambda t}$

We must have $X'(0) = 0 = X'(\pi)$, which is possible only for $\lambda \geq 0$. Then

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$X'(0) = 0 \iff B\sqrt{\lambda} = 0$. Take $B = 0$ or $\lambda = 0$.

Then $X'(\pi) = 0$ only if $\sqrt{\lambda}\pi = n\pi$.

$$\lambda = n^2, \quad n = 0, 1, 2, 3, \ldots$$

$$u(x,t) = B_n e^{-n^2 t}\cos(nx), \quad n = 0, 1, 2, 3, \ldots$$

(Please, do not loose the constant solution

$$u(x,t) = B_0$$

which corresponds to $\lambda = 0$.)

(3b) $u(x,0) = 1 + 7\cos(x)$

By superposition

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-n^2 t} \cos(nx)$$

$$1 + 7\cos(3x) = \sum_{n=0}^{\infty} B_n \cdot 1 \cdot \cos(nx)$$

Thus only the terms with $n=0$ and $n=3$ count:

$$u(x,t) = 1 + 7e^{-9t}\cos(3x)$$

Remark Of course one may use the formulas

$$B_0 = \frac{1}{\pi}\int_0^{\pi} (1 + 7\cos(3x))\,dx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} (1 + 7\cos(3x))\cos(nx)\,dx \quad , \quad n \geq 1$$

for the coefficients in the Fourier cosine series.

④ $$f(z) = \frac{\sin(z)}{z\left(z-\frac{\pi}{2}\right)\left(z+\frac{\pi}{2}\right)}$$

The points $0, \frac{\pi}{2}, -\frac{\pi}{2}$ have to be investigated.

$\underline{z=0}$ The singularity is removable, since

$$\lim_{z \to 0} f(z) = -\frac{4}{\pi^2}$$

This is not a pole.

$\underline{z=\frac{\pi}{2}}$ This is a simple pole with residue

$$\lim_{z \to \frac{\pi}{2}} \frac{\sin(z)}{z\left(z+\frac{\pi}{2}\right)} = \frac{2}{\pi^2}$$

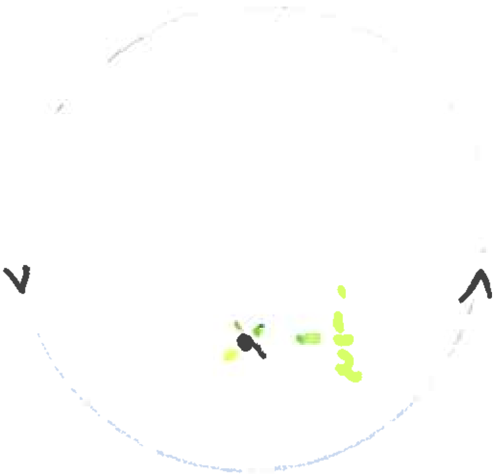
$\underline{z=-\frac{\pi}{2}}$ This is a simple pole with residue

$$\lim_{z \to -\frac{\pi}{2}} \frac{\sin(z)}{z\left(z-\frac{\pi}{2}\right)} = -\frac{2}{\pi^2}$$

(5) $z = e^{i\theta}$; $dz = ie^{i\theta}d\theta$, $\frac{dz}{z} = i\,d\theta$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin(\theta)} = \oint_{|z|=1} \frac{\frac{dz}{iz}}{5 + 4\,\frac{z - \frac{1}{z}}{2i}}$$



$$= \oint_{|z|=1} \frac{dz}{2z^2 + 5iz - 2}$$

$$2z^2 + 5iz - 2 = 0 \iff z = -2i, -\frac{i}{2}$$

Only the pole at $-\frac{i}{2}$ satisfies $|z| < 1$.

$$\operatorname*{Res}_{z=-\frac{i}{2}}\left(\frac{1}{2z^2 + 5iz - 2}\right) = \lim_{z \to -\frac{i}{2}} \frac{z + \frac{i}{2}}{2z^2 + 5iz - 2} = \frac{1}{3i}$$

By the Residue Theorem the integral is

$$= 2\pi i \cdot \frac{1}{3i} = \frac{2\pi}{3}$$

⑥ $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-i\omega x} \sin(3x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-i\omega x} \frac{e^{3ix} - e^{-3ix}}{2i}\, dx$$

$$= \frac{1}{i\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{e^{ix(3-\omega)} - e^{-ix(3+\omega)}}{2}\, dx$$

$$= \frac{1}{i\sqrt{2\pi}} \Bigg/_{-\pi}^{\pi} \frac{e^{ix(3-\omega)}}{2(3-\omega)i} + \frac{e^{-ix(3+\omega)}}{2(3+\omega)i}$$

After substituting $x = \pm\pi$ the answer can be written in many ways, for example

$$\frac{1}{i\sqrt{2\pi}} \left\{ \frac{\sin((3-\omega)\pi)}{3-\omega} - \frac{\sin((3+\omega)\pi)}{3+\omega} \right\},$$

$$\frac{1}{\sqrt{2\pi}} \frac{6i\sin(\omega\pi)}{9-\omega^2}$$