**SYNOPSIS**

**COMPLEX NUMBERS**

$$z = x + iy$$  
($x$ and $y$ are real numbers)

- $x = \text{Re} z$ = the real part of $z$
- $y = \text{Im} z$ = the imaginary part of $z$

The complex numbers obey the following axioms:

\[
\begin{align*}
\text{I} & \quad z_1 = x_1 + iy_1 = x_2 + iy_2 = z_2 \iff x_1 &= x_2 \land y_1 = y_2 \\
\text{II} & \quad z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2) \\
\text{III} & \quad z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2)
\end{align*}
\]

We write $x = x + i0$, $iy = 0 + iy$ (purely imaginary numbers), $0 = 0 + i0$. Also $x + iy = x + yi$.

\[i^2 = -1\]

**Examples**

- $1 + 3i \neq 1,000 + 3i$
- $(1 + 3i) + (-7 + i) = -6 + 4i$
- $(2 - i)(1 + 3i) = 5 + 5i = 5(1 + i)$
\[ Z_1 = 2 + i \]
\[ Z_2 = 1 + 3i \]
\[ Z_1 + Z_2 = 3 + 4i \]
\[ Z_1 \cdot Z_2 = (2+i)(1+3i) = -1 + 7i \]

Triangeln med hörnen 0, 1, \( Z_1 \) är likformig med den triangel som har hörnen 0, \( Z_2 \), \( Z_1 \cdot Z_2 \).
Example
Solve the equation \( z^4 = i \).

\[
(x + iy)^4 = i, \quad x^2 - y^2 + 2xyi = i,
\]
\[
\begin{align*}
  x^2 - y^2 &= 0 \iff x = \pm y \\
  2xy &= 1
\end{align*}
\]

We get \( x, y = \pm \frac{1}{\sqrt{2}} \). Notice that \( x \) and \( y \) must have the same sign, since \( 2xy = 1 > 0 \). We obtain

\[
z = \pm \frac{1 + i}{\sqrt{2}}, \quad \text{or} \quad \sqrt{i} = \pm \frac{1 + i}{\sqrt{2}}.
\]

Remark When solving \( z^2 = a + ib \) it is worth noticing that

\[
(x + iy)^2 = (x^2 - y^2) + 2xyi = a + bi.
\]

\[
\begin{array}{|c|c|}
\hline
z_1 + z_2 &= z_3 + z_4 \\
\hline
\text{commutation} & z_1z_2 = z_2z_1 \\
\hline
\text{distribution} & z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \\
\hline
\text{associativity} & z_1(z_2z_3) = (z_1z_2)z_3 \\
\hline
\end{array}
\]

Example
\( z^5 + z + 1 = (z^2 + z + 1)(z^3 - z^2 + 1) \)

The multiplication can be performed as for real numbers, for instance,

\[
(100 + 10 + 1)(1000 - 100 + 1) = 1000000 + 10 + 1.
\]

\[
(b - a)(b + a) = b^2 - a^2.
\]

\[
z + 0 = z, \quad 1z = z, \quad z + (-z) = 0, \quad zz^{-1} = 1.
\]

\[
1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} \quad (z \neq 1)
\]

Next page!
\( \overline{z} = x - iy \) is the conjugate of \( z = x + iy \)

**Conjugation**

Geometrically, conjugation is a reflection in the real axis.

\[
(\overline{\overline{z}}) = z, \quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1} \overline{z_2}
\]

\[
(\overline{\frac{z_1}{z_2}}) = \frac{\overline{z_1}}{\overline{z_2}}, \quad z \overline{z} = x^2 + y^2
\]

Notice that from \( z = x + iy \), \( \overline{z} = x - iy \) we get

\[
x = \text{Re} z = \frac{z + \overline{z}}{2}, \quad y = \text{Im} z = \frac{z - \overline{z}}{2i}
\]

\[
z \overline{z} = x^2 + y^2
\]

**Example**

\[
\overline{3 + 4i} = 3 - 4i, \quad (1+i)^2 = (1-i)^2 = -2i.
\]

**Division**

\[
\frac{1}{z} = \frac{\overline{z}}{z \overline{z}} = \frac{x - iy}{x^2 + y^2} \quad (z \neq 0)
\]

\[
\frac{z_1}{z_2} = \frac{\overline{z_1} \overline{z_2}}{\overline{z_2} \overline{z_2}} = \cdots \quad (z_2 \neq 0)
\]

\[
\frac{3 + i}{2 - i} = \frac{(3+i)(2+i)}{(2-i)(2+i)} = \frac{5 + 5i}{4 + 1} = 1 + i
\]

**Notation**

\( z^{-1} = \frac{1}{z}, \quad z \neq 0 \)
MODULUS (= absolute value) and ARGUMENT

\[ z = x + iy \]
\[ n = \frac{1}{\sqrt{x^2 + y^2}} \]
\[ n = |z| = \text{the modulus of } z \]
\[ \theta = \text{arg } z = \text{the argument of } z \]

\[ z = x + iy = n (\cos \theta + i \sin \theta) \]

The argument is manyvalued: \( \theta + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots \) will do as well.

**LEMMA** Suppose that \( n_1 > 0 \) and \( n_2 > 0 \).

Then

\[ n_1 (\cos \theta_1 + i \sin \theta_1) = n_2 (\cos \theta_2 + i \sin \theta_2) \iff n_1 = n_2 \ \& \ \theta_1 = \theta_2 + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots \]

**Example**

\[ 1 + i = \sqrt{2} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \]
\[ |1 + i| = \sqrt{2} \]

\[ -1 = 1 (\cos \pi + i \sin \pi) \]
\[ |-1| = 1 \]
**Lemma** \[ |z, z_2| = |z| \cdot |z_2| \]
\[
\arg(z, z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}
\]

**Proof:** \[
z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)
\]
\[
z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)
\]
\[
z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)
\]
\[
= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]
\]

In particular
\[
(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta
\]
\[
(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta
\]
and so on.

\[
(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)
\]

**de Moivre's formula**, \( n = 0, \pm 1, \pm 2, \ldots \)

\[
\frac{1}{z} = \frac{1}{r_2 (\cos \theta_2 + i \sin \theta_2)} = \frac{\cos \theta_2 - i \sin \theta_2}{r_2}
\]
\[
= \frac{\cos(-\theta_2) + i \sin(-\theta_2)}{r_2}
\]
We have \[ \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad \arg \left( \frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2, \quad \text{when } z_2 \neq 0. \]

**Euler's Formula**

\[ \cos \theta + i \sin \theta = e^{i \theta} \]

\[ \begin{cases} 
  e^{i \theta} = \cos \theta + i \sin \theta \\
  e^{-i \theta} = \cos \theta - i \sin \theta
\end{cases} \]

**Euler**

\[ \cos \theta = \frac{e^{i \theta} + e^{-i \theta}}{2}, \quad \sin \theta = \frac{e^{i \theta} - e^{-i \theta}}{2i} \]

In particular,

\[ e^{i \pi} = -1 \]

As we will learn later, the meaning of the formula above is that

\[ 1 + i \pi + \frac{(i \pi)^3}{3!} + \frac{(i \pi)^5}{5!} + \cdots = -1 \quad \text{or} \quad \lim_{n \to \infty} \left(1 + \frac{i \pi}{n}\right)^n = -1. \]

Now de Moivre's formula can be written as

\[ (e^{i \theta})^n = e^{i n \theta} \quad \text{de Moivre} \]

\[ z = n(\cos \theta + i \sin \theta) = n e^{i \theta} \]
Example: \[ 1 + i = \sqrt{2} e^{i \pi/4} \]

\[ e^{i(\theta + 2k\pi)} = e^{i\theta} \quad \text{Period } 2\pi i \]

**Extraction of Roots**

**Definition**

\[ z^n = w \quad \iff \quad z = \sqrt[n]{w} \]

For example, \[ \sqrt[3]{i} = \pm \frac{1+i}{\sqrt{2}}; \quad \sqrt{1} = \pm 1; \quad \sqrt[4]{1} = 1, -1, i, -i. \]

\[ w = se^{i\gamma} \quad (s > 0), \quad z = ne^{i\theta} \]

\[ (ne^{i\theta})^n = se^{i\gamma}, \]

\[ n^ne^{in\theta} = se^{i\gamma}, \]

\[ n^n = s \quad \text{and} \quad n\theta = \gamma + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots), \]

\[ \begin{cases} \sqrt[n]{s} \quad \text{(the real root, a positive number)}, \\ \frac{\pi}{n} + \frac{2k\pi}{n}. \end{cases} \]

\[ \sqrt[n]{w} = \sqrt[n]{s} e^{i \left( \frac{\pi}{n} + \frac{2k\pi}{n} \right)} \quad (k = 0, \pm 1, \pm 2, \ldots) \]

The \( n \)-th root has exactly \( n \) different values, since the exponential has the period \( 2\pi i \). The choice \( k = 0, 1, 2, 3, \ldots, n-1 \) for instance, yields all the roots.
Rötterna
\[ \sqrt[n]{1} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad (k = 0, 1, 2, \ldots, n) \]
till ekvationen \( z^n = 1 \) bildar en reguljär \( n \)-hörning. Enligt Gauss kan denna polygon konstrueras med passare och linjal om och endast om
\[ n = 2^m p_1 p_2 \cdots p_k, \quad \text{EX.} \, 4 \cdot 5 \cdot 257 \]
där \( m = 0, 1, 2, \ldots \) och \( p_k \) och \( p \) är olika primitiv av formen
\[ p = 2^{(2^m)} + 1 \quad (\text{Fermats primital}) \]

Fermats Primital: 3, 5, 17, 257, 65537,

\[ 2^5 + 1 = 641 \cdot 6700417 \quad (\text{Euler 1732}) \]

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 51, ...
Example

\[ \sqrt[3]{8i} = 2 \]

\[ z^3 = 8i, \quad (ne^{i\theta})^3 = 8e^{i\frac{\pi}{2}}, \quad n^3 \cdot e^{3i\theta} = 8e^{i\frac{\pi}{2}} \]

\[ \begin{cases} n^3 = 8, & n = 2 \\ 3\theta = \frac{\pi}{2} + 2k\pi \end{cases} \]

Hence, \( n = 2 \) and \( \theta = \frac{\pi}{6} + \frac{2k\pi}{3} \), \( k = 0, \pm 1, \pm 2, \ldots \)

\[ z = \begin{cases} 2e^{i\pi/6} = \sqrt{3} + i & k = 0 \\ 2e^{i5\pi/6} = -\sqrt{3} + i & k = 1 \\ 2e^{i3\pi/2} = -2i & k = 2 \end{cases} \]

The 3rd root has exactly 3 different values.

\[ \text{THE TRIANGLE INEQUALITIES} \]

\[ \pm (|z_1| - |z_2|) \leq |z_1 + z_2| \leq |z_1| + |z_2| \]

\[ \begin{array}{cc}
\text{The Binomial Formula} \\
(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \\
+ \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n \\
\end{array} \]

The coefficient in front of \( a^{n-k}b^k \) is \( \binom{n}{k} \)

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} \]

from Pascal's triangle:

\[ \begin{array}{ccccccc}
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 \\
\end{array} \]
POLYNOMIALS, ALGEBRAIC EQUATIONS, THE FUNDAMENTAL THEOREM OF ALGEBRA

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0 \]

\[
P(z) - P(z_1) = a_1 (z-z_1) + a_2 (z^2-z_1^2) + \cdots + a_n (z^n-z_1^n)
\]

\[
= (z-z_1) \left\{ a_1 + a_2 (z+z_1) + a_3 (z^2+z_1z_1+z_1^2) + \cdots + a_n (z^{n-1}+z_1^{n-2}z_1+z_1^{n-1}) \right\}
\]

\[
= (z-z_1) \cdot \{"a \text{ polynomial}"\}. \quad \text{This means that}
\]

if \( P(z_1) = 0 \), then \( P(z) \) contains the factor \( z-z_1 \).

**Example**

\[ P(z) = z^3-z^2+z-1. \quad \text{We see that} \]

\[ P(1) = 0. \quad \text{Indeed, dividing by} \ z-1, \]

\[
\frac{z^3-z^2+z-1}{z-1} = z^2 + 1.
\]

Finally,

\[
z^3-z^2+z-1 = (z-1)(z+i)(z-i)
\]

**Lemma**

Suppose that the coefficients \( a_0, a_1, a_2, \ldots, a_n \) are real numbers. If \( z \) is a root of the equation

\[ a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \]

so is the conjugate number \( \overline{z} \).

\[ a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a b^{n-2} + b^{n-1}) \]

Multiply to see this!
\[ n^0: \quad 0 = \overline{0} = a_n \overline{z}^n + \cdots + a_2 \overline{z}^2 + a_1 \overline{z} + a_0 \]
\[ = a_n \overline{z}^n + \cdots + a_2 \overline{z}^2 + a_1 \overline{z} + a_0 = \overline{a_n \overline{z}^n + \cdots + a_2 \overline{z}^2 + a_1 \overline{z} + a_0} = \overline{P(z)}, \]

since \( a_k = \overline{a}_k \) by assumption.

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\[ a = d + i\beta, \quad \beta \neq 0 \]
\[ (z - a)(z - \overline{a}) = z^2 - 2dz + (d^2 + \beta^2) = |a|^2 \]

A quadratic factor without real roots.

The conjugate roots combine into a quadratic factor with no real zeros.

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**Example** \( z^4 + 1 = 0 \)

\[ z^4 + 1 = (z - e^{i\pi/4})(z - e^{-i\pi/4})(z - e^{3i\pi/4})(z - e^{-3i\pi/4}) \]

\[ = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1) \]

The factorization \( x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1) \) can be used for the evaluation of the integral \( \int \frac{dx}{x^4 + 1} \).

\[ z^4 + 1 = \left( z - \frac{1+4i}{\sqrt{2}} \right) \left( z - \frac{-1-4i}{\sqrt{2}} \right) \left( z + \frac{1+4i}{\sqrt{2}} \right) \left( z + \frac{-1-4i}{\sqrt{2}} \right) \]

The equation \( z^4 - 1 = 0 \) is simpler: \( \pm 1, \pm i \).
The Fundamental Theorem of Algebra was proved by Gauss in 1799. It states that an equation of the $n^{th}$ order always has exactly $n$ complex roots. As a consequence

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = (z-z_1)(z-z_2)\cdots(z-z_n)$$

The same root may be repeated, for example

$$z^3 - 3z^2 + 3z - 1 = (z-1)(z-1)(z-1),$$

that is, $z=1$ is a triple root of $z^3 - 3z^2 + 3z - 1 = 0$.

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**Some Geometry**

- **Disc**: $|z| < 2$
- **Annulus**: $1 < |z| < 2$
- **Half-plane**: $\text{Re}[z(i-1)] > 1$
- **Punctured disc**: $0 < |z - z_0| < \rho$
  - Radius $\rho$
  - Centre $z_0$

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**Bernoulli's Lemniscate**

$$|z-1||z+1| = 1$$
ANALYTIC FUNCTIONS

\[ f(z) = u(x, y) + i \nu(x, y) \]

**Example** \( f(z) = z^3 \). In this example

\[
\begin{align*}
  f(z) &= z^3 = (x + iy)^3 = x^3 - 3x^2y + 3xy^2 + i(3x^2y - y^3) \\
  &= u(x, y) + i \nu(x, y)
\end{align*}
\]

We can also write \( f(re^{i\theta}) = r^3 e^{3i\theta}, \ z = re^{i\theta} \).

**DEF.** Suppose that \( f(z) \) is defined in the punctured disc \( 0 < |z - a| < \delta \). We say that \( f(z) \to A, \) as \( z \to a, \) if

\[
\lim_{z \to a} |f(z) - A| = 0.
\]

In other words, given \( \varepsilon > 0, \) there is a \( \delta_\varepsilon > 0 \) such that

\[
|f(z) - A| < \varepsilon, \ \text{when} \ 0 < |z - a| < \delta_\varepsilon.
\]

**NOTATION** \( \lim_{z \to a} f(z) = A \iff \lim_{z \to a} |f(z) - A| = 0 \)

We also write \( \lim_{z \to a} f(z) = \infty, \) when \( \lim_{z \to a} |f(z)| = \infty. \)

**Example** \( \lim_{z \to i} \frac{z^3 + 1}{z - i} = 2i \). This is evident, since \( \frac{z^3 + 1}{z - i} = z + i, \) when \( z \neq i. \)** **NOTICE THAT** \( z \) **MAY APPROACH \( i \) FROM ANY DIRECTION!**
Example \( f(z) = \frac{z}{|z|} \), \( z \neq 0 \).

\[
\lim_{x \to 0^+} f(x) = 1, \quad \lim_{y \to 0^+} f(iy) = i, \quad \lim_{x \to 0^-} f(x) = -1.
\]

The values 1, i, -1 depend on the direction from which the origin is approached. Hence \( f(z) \) does not have a limit, as \( z \to 0 \).

**Usual rules for limits of expressions like**

\[
f(z) \pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)}, \quad F(f(z))
\]

are valid.

**DEF.** Suppose that \( f(z) \) is defined in the disc \( |z - a| < R \). We say that \( f(z) \) is **continuous at the point** \( a \), if

\[
\lim_{z \to a} f(z) = f(a).
\]

Roughly, this means that \( f(z) \approx f(a) \), when \( z \approx a \). A small error in \( a \) (more exactly, a sufficiently small error in \( a \)) leads to only a small error for \( f(a) \).

Example

\[
g(z) = \begin{cases} 
\frac{z^2 + 1}{z - i}, & \text{when } z \neq i, \\
i, & \text{when } z = i.
\end{cases}
\]

The function \( g(z) \) is continuous at every point.
**Lemma**  \( \lim f(z) = A \iff \lim \Re f(z) = \Re A \land \lim \Im f(z) = \Im A \)

**Proof:** To see this, write  
\[
\{ = \mu + i \upsilon, \quad A = a + i b. \\
\]

\[
|f(z) - A| = \sqrt{(\mu(x,y) - a)^2 + (\upsilon(x,y) - b)^2}. \\
\]

\[
\frac{d f}{d z} = f'. \\
\]

**The Derivative**

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
\]

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
\]

It is important that \( \Delta z = \Delta x + i \Delta y \) may approach the origin from any direction.

Given \( \varepsilon > 0 \), there is a \( \delta_\varepsilon > 0 \) such that  
\[
|f'(z) - \frac{f(z + \Delta z) - f(z)}{\Delta z}| < \varepsilon, \text{ when } 0 < |\Delta z| < \delta_\varepsilon. \\
\]

Here  
\[
|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}. \\
\]

**Ex:**  
\[
f(z) = z^2, \quad \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2z + \Delta z \to 2z; \quad \frac{d z^2}{d z} = 2z. \\
\]
\[ f(z) = \overline{z} = x - iy \]

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{z + \Delta z - \overline{z}}{\Delta z} = \frac{\Delta z}{\Delta z} \]

\[ = \frac{\Delta x + i \Delta y}{\Delta x + i \Delta y} = \frac{\Delta x - i \Delta y}{\Delta x + i \Delta y} \]

\[ \lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = 1 \neq -1 = \lim_{i \Delta y \to 0} \frac{f(z + i \Delta y) - f(z)}{i \Delta y} \]

Hence \( f(z) = \overline{z} \) is not differentiable at any point.

As a rule of thumb, a differentiable function is a function only of the variable \( z \) but not of \( \overline{z} \) or \( |z| \) (notice \( |z|^2 = \overline{z}z \)).

**Differentiation Rules:**

\[ (f + g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left( \frac{1}{g} \right)' = \frac{-g'}{g^2} \]

\[ \frac{d}{dz} f(g(z)) = \left( \frac{df(\xi)}{d\xi} \right) \frac{dg(z)}{dz} \]

A differentiable function is continuous.
DEFINITION We say that the function \( f(z) \) is analytic in the domain \( \Omega \), if the derivative \( f'(z) \) exists, when \( z \in \Omega \).

\[ \text{analytic} = \text{holomorphic}; \ \text{conformal mapping} = \text{analytic function whose derivative is different from 0} \]

Since \( z \) is analytic, so is \( z^3 = z \cdot z \) and so is \( z^3 = z^2 \cdot z \) (recall that \( (fg)' = fg' + fg' \)) and...

In general, \( z^n \) is analytic in the whole complex plane and

\[ \frac{dz^n}{dz} = nz^{n-1}. \quad (n = 1, 2, 3, \ldots) \]

A polynomial \( a_0 + a_1z + \cdots + a_nz^n \) is analytic in the whole complex plane. A rational function

\[ \frac{a_0 + a_1z + a_2z^2 + \cdots + a_nz^n}{b_0 + b_1z + b_2z^2 + \cdots + b_mz^m} \]

is analytic, when the points where the denominator is zero (the so-called poles) are excluded.

As we will see later, the functions \( e^z, \sin z, \cos z, \tan z, \cot z, \sinh z, \cosh z, \tanh z, \coth z, \ln z, \arcsin z, \ldots \) are analytic. One of the miracles about analytic functions is that not only does \( f'(z) \) exist but also all higher derivatives \( f''(z), f'''(z), f^{(4)}(z), \ldots \) exist!
CAUCHY–RIEMANN EQUATIONS

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

THEOREM  If \( f = u + iv \) is analytic in the domain \( \Omega \), then \( u \) and \( v \) satisfy the Cauchy–Riemann equations in \( \Omega \).

Proof: Fix \( z = x + iy \) in \( \Omega \). Let us calculate \( f'(z) \) in two ways. Recall that \( \Delta z = \Delta x + i \Delta y \) is allowed to approach 0 in any way. First, let \( \Delta z = \Delta x \). Then

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \]

\[ \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} \to \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \]

Thus

\[ f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \]

This formula is useful.

Second, let \( \Delta z = i \Delta y \). Then

\[ \frac{f(z + i \Delta y) - f(z)}{i \Delta y} = \frac{u(x, y + \Delta y) - u(x, y)}{i \Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i \Delta y} \]
\[ \frac{\partial u}{\partial y} \rightarrow 0 \quad \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} . \]

Thus

\[ f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} . \]

Equating the two expressions for \( f'(z) \) we have

\[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} . \]

This yields the Cauchy-Riemann equations.

Example \[ f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} , \quad z = x + iy \neq 0 . \]

The functions \[ u = \frac{x}{x^2 + y^2} \quad , \quad v = -\frac{y}{x^2 + y^2} \]

satisfy the Cauchy-Riemann equations. Indeed

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y} , \\
\frac{\partial u}{\partial y} &= -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x} .
\end{align*}
\]

We can also verify that

\[ f'(z) = -\frac{1}{z^2} = -\frac{\overline{z}}{|z|^2} = -\frac{(x - iy)}{(x^2 + y^2)^2} \]

\[ = \frac{y^2 - x^2 + 2ixy}{(x^2 + y^2)^2} = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} . \]
Example Suppose that $f(z)$ is analytic and that $|f(z)|$ is constant. Then $f(z)$ is constant.

\[
\begin{align*}
\begin{cases}
u = u + iv, & u^2 + v^2 \equiv C^2 \text{ (a constant)} \quad \text{不容易理解的一部分} \\
2u \mu_x + 2v \nu_x &= 0 \\
2u \mu_y + 2v \nu_y &= 0
\end{cases}
\end{align*}
\]

After differentiation. Here $\mu_x = \frac{\partial u}{\partial x}$, ... 

(1) $\begin{cases}
\mu u_x - \nu u_y = 0 \\
\mu u_y + \nu u_x = 0
\end{cases}$ By the Cauchy–Riemann equations

(2) $u^2 u_x = \mu \nu u_y = -v^2 u_x$, $(u^2 + v^2) u_x = 0$

We get $(u^2 + v^2) u_x = 0$, $(u^2 + v^2) u_y = 0$.

If $C = 0$, $u = v = 0$ and $f(z) \equiv 0$. If $C \neq 0$, then $u_x \equiv 0$ and $u_y \equiv 0$ and so $u \equiv \text{ constant}$. By the Cauchy–Riemann equations $v_x \equiv 0$ and $v_y \equiv 0$ and so $v \equiv \text{ constant}$. This proves that $f(z)$ is a constant.

If the Cauchy–Riemann equations hold, then we have an analytic function. More precisely:

**THEOREM** Suppose that the first partial derivatives $\mu_x, \mu_y, \nu_x, \nu_y$ are continuous and satisfy the Cauchy–Riemann equations $\mu_x = \nu_y, \mu_y = -\nu_x$ in the domain $\Omega$. Then the function $f = u + iv$ is analytic in $\Omega$.

**Proof:** Skipped!
SOME ANALYTIC FUNCTIONS

\[ e^z = e^x (\cos y + i \sin y), \quad z = x + iy \]

\[ \frac{de^z}{dz} = e^z, \quad |e^z| = e^x, \quad e^z \neq 0 \]

\[ e^{z_1 + z_2} = e^{z_1} e^{z_2} \]

\[ e^z = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n \quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \]

\[ e^{z + 2\pi i} = e^z \quad (\text{period} \ 2\pi i) \]

\[ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \]

\[ \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z} \]

\[ \sin^2 z + \cos^2 z = 1 \]

\[ \begin{cases} 
\sin(z + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1 \\
\cos(z + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 
\end{cases} \]

\[ \sin(z + 2\pi) = \sin z \quad (\text{period} \ 2\pi) \]

| \sin x | \leq 1 \quad \text{but it is possible that} \ |\sin z| > 1 \quad !

\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \]

\[ \pi^{\sqrt{3}} = 2.62537412640768793 \cdots \]

\[ \frac{\pi}{\sqrt{3}} = 2.34781536736726085 \cdots \]

\[ 3 + 4 + 3 + 5 + \cdots \]
\[ \sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2} \]

\[ \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{2z} - 1}{e^{2z} + 1}, \quad \coth z \]

\[ \cosh^2 z - \sinh^2 z = 1 \]

\[ \frac{d\cosh z}{dz} = \sinh z \]

\[ \cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z, \]
\[ \cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z \]

**Example:** Starting with the familiar formula

\[ \sin^2 z + \cos^2 z = 1 \]

we get, replacing \( z \) by \( iz \),

\[ \sinh^2(iz) + \cosh^2(iz) = 1, \quad (i \sinh z)^2 + (\cosh z)^2 \]

that is \( \cosh^2 z - \sinh^2 z = 1 \).

---

\[ e^w = z, \quad z \neq 0 \quad \text{DEF.} \quad w = \ln z \]

**The notation** \( \log z \) \( \text{is frequently used.} \)

**Preliminary remark:** if \( e^w = z \), then \( e^{w+2\pi ni} = z \). Thus \( \ln z \) is defined only up to a multiple of \( 2\pi i \).

\[ w = u + iv, \quad z = re^{i\theta} \]

\[ e^w = z \quad \iff \quad e^u e^{iv} = re^{i\theta} \quad \iff \quad u = \ln r \quad \text{and} \quad v = \theta + 2\pi n \]

\[ w = \ln r + i(\theta + 2\pi n), \quad n = 0, \pm 1, \pm 2, ... \]
\[ \ln z = \ln |z| + i \arg(z) \]

**Examples**

\[ \ln(-1) = i (\pi + 2n\pi) \]

\[ \ln(i) = i \left( \frac{\pi}{2} + 2n\pi \right) \]

\[ \ln 1 = 0 + 2n\pi i \quad \text{Usually, we take } n = 0. \]

\[ \ln(z_1 z_2) = \ln z_1 + \ln z_2 \quad \text{(modulo } 2\pi i) \]

**DEF.** \[ a^z = e^{z \ln a} \quad (a \neq 0) \]

**Examples**

\[ i^i = e^{i \ln i} = e^{-\frac{\pi}{2}} e^{2n\pi} \]

\[ \sqrt{z} = z^\frac{1}{2} = e^{\frac{1}{2} \ln z} \]

**Remark** The only analytic function which coincides with \( e^x \) on the real axis is \( e^z \). No other function will do. More generally, we have the **Principle of Analytic Continuation**: Suppose that the functions \( f(z) \) and \( g(z) \) are analytic and that \( f(x) = g(x) \) on the real axis. Then \( f(z) = g(z) \) (in the domain of analyticity).

\[ 1 = (-1)^2 = \sqrt{(-1)^2} = \sqrt{-1 \cdot -1} = \sqrt{-1} \sqrt{-1} = i \cdot i = -1 \]

\[ 2 \log(-1) = \log(-1) + \log(-1) = \log[-1 \cdot -1] = \log 1 = 0 \Rightarrow \]

\[ \log(-1) = 0 = \Rightarrow e^0 = -1. \quad \text{But } e^0 = +1, \text{ as we know!} \]
The complex exponential function

\[ e^z = \exp z = e^x (\cos y + i \sin y) \]

is periodic with \(2\pi i\), reduces to \(e^x\) if \(z = x \ (y = 0)\), and has the derivative \(e^z\).

The trigonometric functions are

\[ \cos z = \frac{1}{2} (e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \]
\[ \sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \]

\[ \tan z = (\sin z)/\cos z, \ \cot z = 1/\tan z, \ \text{etc.} \]

The hyperbolic functions are

\[ \cosh z = \frac{1}{2} (e^z + e^{-z}) = \cos iz, \]
\[ \sinh z = \frac{1}{2} (e^z - e^{-z}) = -i \sin iz, \]

etc. An entire function is a function that is analytic everywhere in the complex plane. The functions in (5)–(7) are entire.

The natural logarithm is

\[ \ln z = \ln |z| + i \arg z \quad (\arg z = \theta, \ z \neq 0) \]
\[ = \ln |z| + i \text{Arg} \ z \pm 2n\pi i \quad (n = 0, 1, \cdots), \]

where \(\text{Arg} \ z\) is the principal value of \(\arg z\), that is, \(-\pi < \text{Arg} \ z \leq \pi\). We see that \(\ln z\) is infinitely many-valued. Taking \(n = 0\) gives the principal value \(\text{Ln} \ z\) of \(\ln z\); thus

\[ \text{Ln} z = \ln |z| + i \text{Arg} \ z. \]

General powers are defined by

\[ z^c = e^{c \ln z} \quad (c \ \text{complex}, \ z \neq 0). \]
**LAPLACE's EQUATION** \( \Delta u = 0 \)

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\]

The real and imaginary parts of an analytic function satisfy the Laplace equation:

\[
\{ u + iv ; \Delta u = 0, \Delta v = 0 \}
\]

This follows from the Cauchy - Riemann equations, if we already know that \( u \) and \( v \) has continuous second partial derivatives.

The solutions of Laplace's equation are called harmonic functions. They are important in potential theory (gravitation, electricity, ...).

**Ex.:** \( \Delta \left( \frac{x}{x^2 + y^2} \right) = 0 \), when \((x, y) \neq (0, 0)\). \( \Re \frac{1}{z} \) !

If \( \Delta u = 0 \), we can construct (at least locally) a function \( v \) such that \( u + iv \) is analytic. (We say that \( u \) and \( v \) are conjugate harmonic functions.)

**EXAMPLE** \( u(x, y) = 2x(1-y) \). \( v(x, y) = ? \)

\[
\Delta u = u_{xx} + u_{yy} = 0 \quad \text{Hence } u \text{ is harmonic.}
\]

**Cauchy-Riemann**

\[
\begin{align*}
\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} = 2(1-y) & \iff & u &= 2y - y^3 + \varphi(x) \\
\frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y} = 2x & \quad & v_x = 0 + \varphi'(x)
\end{align*}
\]

Hence \( 2x = \varphi'(x) \), \( \varphi(x) = x^2 + C \). Thus \( v = 2y + x^2 - y^3 + C \)

As a matter of fact

\( \{ iz \} = u + iv = i z^2 + 2z + iC \)

**Answer**
**INTEGRALS**

Curves \( Z(t) = x(t) + iy(t) \), \( t = \text{"time"} \)

\[
\dot{Z}(t) = \frac{dZ}{dt} = \lim_{\Delta t \to 0} \frac{Z(t+\Delta t) - Z(t)}{\Delta t}
\]

\[
= \ldots = \dot{x}(t) + i \dot{y}(t)
\]

\[
|\dot{Z}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \quad \text{"SPEED"}
\]

Length of the curve = \( \int_{t_0}^{t_\infty} |\dot{Z}(t)| \, dt \)

---

**Examples**

1) \( Z = 1 + ti \), \( 0 \leq t \leq 1 \)

\( \dot{Z} = i \) A segment.

2) \( Z(t) = 2 \cos t + 2i \sin t = 2e^{it} \)

\( \dot{Z}(t) = -2i \sin t + 2i \cos t = 2ie^{it} \)

\( x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4 \) A circle of radius 2.

3) \( Z(t) = t + it^2 \), \( -\infty < t < \infty \)

\( Z_a(t) = \ln t + i(\ln t)^2 \), \( 0 < t < \infty \)

Both represent the same curve, the parabola \( y = x^2 \).
Curve integrals (line integrals)

\[ \int_C f(z) \, dz \]

\( f(z) \) is a given function.

\[ \Delta z_j = z_j - z_{j-1} \]

1°) Pick arbitrary points \( a = z_0, z_1, z_2, z_3, \ldots, z_{n-1}, z_n = b \) in successive order, dividing the curve into \( n \) arcs, the so-called subarcs.

2°) Select a point \( z_j \) from the arc between \( z_{j-1} \) and \( z_j \), \( j = 1, 2, 3, \ldots, n \)

3°) Form the sum (Riemann's sum)

\[ \sum_{j=1}^{n} f(z_j) \Delta z_j \]

Suppose that the division is done so that the length of the longest subarc (and hence the lengths of all subarcs) approaches zero. Under suitable assumptions the Riemann sums will approach a limit.
that is independent of the division:
\[
\lims \sum \xi_j \Delta z_j = \int_C \xi(z) \, dz
\]
This is the integral of \( \xi(z) \) along the curve \( C \).

If the curve \( C \) has the parametrization
\[
z = z(t) = x(t) + iy(t), \quad a \leq t \leq \beta,
\]
and if \( \ddot{z}(t) \) is continuous, we can use the formula
\[
\int_C \xi(z) \, dz = \int_a^\beta \xi(z(t)) \dot{z}(t) \, dt
\]
for the calculation. (As a rule of thumb, the integrand \( \xi(z(t)) \dot{z}(t) \) should be continuous.)

**Ex.**
\[
\int_0^{2\pi} \frac{dz}{z} = 2\pi i
\]
The unit circle, counterclockwise.

\( z = e^{i t}, \quad dz = i e^{i t} \, dt, \quad \frac{dz}{z} = \frac{i e^{i t} \, dt}{e^{i t}} = i \, dt \)
\[
\oint \frac{dz}{z} = \int_0^{2\pi} i \, dt = 2\pi i. \quad \text{A similar calculation yields}
\]
\[
\oint z^n \, dz = \begin{cases} 
2\pi i, & n = -1 \\
0, & n = 0, 1, \pm 2, \pm 3, \ldots 
\end{cases}
\]
again counterclockwise along the unit circle \(|z| = 1|\).
**Interpretation of** $\oint \frac{dz}{z} = 2\pi i$.

$$2\pi i = \oint \frac{dx + idy}{x + iy} = \oint \frac{x\, dx + y\, dy}{x^2 + y^2} = 0$$

The work done by the force $\overline{F} = x\hat{i} + y\hat{j}$ is 0 and the work done by the force $\overline{G} = -y\hat{i} + x\hat{j}$ is $2\pi i$. (Notice that $x^2 + y^2 = 1$.)

---

**Example** The integral $\int_{0}^{2} \overline{z} \, dz$ from 0 to 2 along the half-circle is $2-i\pi$, and along the line segment it is 2. Indeed,

I) $z(t) = 1 + e^{it}$, half-circle

$$dz = ie^{it} \, dt$$

$$\int_{0}^{\pi} (1 + e^{-it}) ie^{it} \, dt = \int_{0}^{\pi} i [e^{it} + 1] \, dt$$

$$= \int_{0}^{\pi} e^{it} + it = 2 - i\pi$$

II) $z = x$, $dz = dx$; $\overline{z} = x$ line segment

$$\int_{0}^{2} \overline{z} \, dz = \int_{0}^{2} x \, dx = \int_{0}^{2} \frac{x^2}{2} = 2$$
The ML-inequality

$$\left| \int_{C} f(z) \, dz \right| \leq ML$$

$$L = \int_{C} |dz| = \text{length of the curve } C$$

$$|f(z)| \leq M, \text{ when } z \in C$$

is useful. (The function $f(z)$ does not have to be analytic here.)

Example: Show that the integral

$$\int_{C} \frac{e^{3iz}}{z^2 + 2z + 10} \, dz$$

taken along the arc $z = R e^{i\theta}$, $0 \leq \theta \leq \pi$, of a half-circle approaches zero as $R \to +\infty$.

$$e^{3iz} = e^{3i(R \cos \theta + iR \sin \theta)} = e^{-3R \sin \theta} e^{3iR \cos \theta}$$

$$\left| e^{3iz} \right| = e^{-3R \sin \theta} \leq 1, \text{ when } 0 \leq \theta \leq \pi.$$ 

$$|z^2 + 2z + 10| \geq |z^2 - 12z + 10| \geq R^2 - 2R - 10$$

$$\left| \frac{e^{3iz}}{z^2 + 2z + 10} \right| \leq \frac{1}{R^2 - 2R - 10}.$$  

By ML-ineq.

$$\left| \int_{C} dz \right| \leq \frac{\pi R}{R^2 - 2R - 10} \to 0, \text{ as } R \to +\infty.$$
Integrals of analytic functions in simply connected domains can be evaluated in the following way.

**THEOREM** Let \( f(z) \) be analytic in a simply connected domain \( \Omega \). Then there exists a function \( F(z) \) such that \( F'(z) = f(z) \) when \( z \in \Omega \). Moreover,

\[
\int_{z_1}^{z_2} f(z) \, dz = F(z_2) - F(z_1)
\]

where the integral is taken along any path in \( \Omega \) joining \( z_1 \) and \( z_2 \).

The integral depends only on the endpoints \( z_1 \) and \( z_2 \) but is independent of the path. The proof of the theorem is based on Cauchy's Integral Theorem, a fundamental result, which will be derived later.

**EXAMPLES**

\[
\int_{0}^{1+i} z^2 \, dz = \int_{0}^{1+i} \frac{z^3}{3} = \left. \frac{(1+i)^3}{3} \right|_0 = \frac{2i-2}{3}
\]

\[
\int_{z_1}^{z_2} \sin z \, dz = -\cos z_2 + \cos z_1
\]
EXAMPLE  The integral \( \int_{-i}^{i} \frac{dz}{z} \) depends on the path between the endpoints \( \pm i \).

Direct calculation (write \( z = e^{i\theta}, \; dz = i e^{i\theta} \, d\theta \) and so on) yields

\[
\int_{-i}^{i} \frac{dz}{z} = \begin{cases} \pi i & \text{along the right half-circle} \\ -\pi i & \text{along the left half-circle} \end{cases}
\]

Hence the formula

\[
\int_{-i}^{i} \frac{dz}{z} = \ln i - \ln(-i)
\]

Recall that

\[
\frac{d\ln z}{dz} = \frac{1}{z}
\]

cannot be used without some precaution.

In general, we have to specify a simply connected domain not containing the origin (\( 1/z \) is analytic except at the origin) but containing the desired half-circle. For instance, removing the negative real axis (a slit from 0 to \(-\infty\)), a simply connected domain remains.

\[
\ln z = \ln n + i \theta + 2\pi i n, \quad n = 0
\]

Now \( \int_{-i}^{i} \frac{dz}{z} = \ln i - \ln(-i) = i \frac{\pi}{2} - (-i \frac{\pi}{2}) = i\pi \)

\(-\pi < \theta < +\pi\) along any path not crossing the slit \((-\infty, 0]\). In particular, \( \int_{-\infty}^{0} \frac{dz}{z} \)
the right (not the left) half-circle.

REMARK. Usually, the functions $\ln z^2$ and $\sqrt{z}$ require a slit from zero to infinity. This prevents winding around 0.

---

CAUCHY'S INTEGRAL THEOREM

THEOREM If $f(z)$ is analytic in a simply connected domain $\Omega$, then for every closed path (= "contour") in $\Omega$

$$\oint f(z)dz = 0.$$ 

Ex. $\oint e^zdz = 0$, $\oint \sin z\,dz = 0$, $\oint z^3\,dz = 0$, $\oint e^{\sin z}\,dz = 0$. The contour is arbitrary.

Ex. $\oint \frac{dz}{1+z^2} = 0$ provided that the curve does not enclose $±i$. 

---

---
A precise calculation (later!) reveals that the integral

\[ \oint \frac{dz}{1 + z^2} = \begin{cases} \pi & \text{along } C_1 \\ -\pi & \text{along } C_2 \\ 0 & \text{along } C_3 \\ 0 & \text{along } C_4 \end{cases} \]

Cauchy's Integral Theorem can be directly applied only for the path \( C_4 \).

---

Proof of Cauchy's Integral Theorem under the additional assumption that \( f'(z) \) is continuous. Recall Green's formula

\[ \oint_C (P \, dx + Q \, dy) = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy \]

Here \( D \) is the domain bounded by \( C \). We have

\[ \oint_C f(z) \, dz = \oint_C (u + iv) \, (dx + i \, dy) = \oint_C \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \, dy \]

\[ + i \oint_C \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \, dx \, dy \]

\[ = \iint_D \left( \frac{\partial (u + iv)}{\partial x} - \frac{\partial (u + iv)}{\partial y} \right) dx \, dy \]

\[ + i \iint_D \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) dx \, dy \]

\[ = \iint_D \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \, dy \]

\[ + i \iint_D \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} dx \, dy \]

\[ = \iint_D \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \, dy \]

\[ + i \iint_D \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} dx \, dy \]

\[ = \iint_D \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \, dy \]
\[ + i \int \int_D \partial x \partial y = 0 + i \cdot 0, \text{ where the} \]

Cauchy–Riemann equations were used:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0. \]

\[ \square \]

**Remark.** About 1900 Goursat removed the assumption about the continuity of the derivative \( f'(z) \). He merely assumed that \( f'(z) \) exists, i.e., that \( f(z) \) is analytic, in his proof. One often calls the result the **Cauchy–Goursat Theorem**.

**Principle of Deformation of Path.** In multiply connected domains this is of actual interest.

If the closed curve \( C_1 \) can be "continuously deformed" into \( C_2 \) without even touching the boundary of the domain of analyticity for \( f(z) \), then

\[ \int_{C_1} f(z)dz = \int_{C_2} f(z)dz. \]
For example
\[ \oint_C \frac{dz}{z-z_0} = \begin{cases} \frac{2\pi i}{z_0}, & \text{if } C \text{ encloses } z_0 \\ 0, & \text{if } z_0 \text{ is outside the domain bounded by } C. \end{cases} \]

If \( C \) encloses \( z_0 \), then we can deform \( C \) to a circle centered at \( z_0 \).

Thus
\[ \oint_C \frac{dz}{z-z_0} = \oint_{\text{Circle} \ 1z-201=1} \frac{dz}{z-z_0} = \frac{2\pi i}{z_0} \]

The other case follows from Cauchy's theorem:

Now \( \frac{1}{z-z_0} \) is analytic in the domain bounded by \( C \), indeed.

The idea is:

Because of cancellations and reversions of direction
\[ \oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz \]

Note: \( f(z) \) is analytic in the domain between the curves \( C_1 \) and \( C_2 \). The segments "---" divide this domain into two simply-connected domains, where Cauchy's Int. Theorem can be applied.
CAUCHY'S INTEGRAL FORMULA:

\[ f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) \, dz}{z-z_0} \]

Let \( f(z) \) be analytic in a simply connected domain \( \Omega \). Suppose that \( z_0 \) is a point in \( \Omega \). If the closed curve \( C \) lies in \( \Omega \) and encloses \( z_0 \), then the formula above holds.

Example:

\[ 1 = e^0 = \frac{1}{2\pi i} \oint_C \frac{e^z \, dz}{z-0} \]

Proof: The integrand \( \frac{f(z)}{z-z_0} \) is analytic except at the point \( z_0 \). Deforming the path \( C \) to a small circle \( |z-z_0| = \rho \), we get

\[ \oint_C \frac{f(z) \, dz}{z-z_0} = \left[ \oint_C \frac{f(z) \, dz}{z-z_0} \right]_{|z-z_0|=\rho} = 0 \]
Writing
\[ f(z) = f(z_0) + \left[ f(z) - f(z_0) \right] \]
we arrive at
\[ \oint_C \frac{f(z)}{z-z_0} \, dz = f(z_0) \oint_C \frac{dz}{z-z_0} + \oint_C \frac{f(z) - f(z_0)}{z-z_0} \, dz \]
\[ = 2\pi i f(z_0) \]

As a matter of fact, the second integral is zero. Indeed, given an arbitrarily small number \( \varepsilon > 0 \), \( |f(z) - f(z_0)| < \varepsilon \) when \( |z - z_0| = \delta \) is small enough. (This is the meaning of \( f(z_0) = \lim_{z \to z_0} f(z) \).) Then
\[ \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \frac{\varepsilon}{\delta} \]
and
\[ \left| \oint_C \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \frac{\varepsilon}{\delta} \cdot 2\pi \delta = 2\pi \varepsilon. \]

But \( \varepsilon > 0 \) is as small as we please. Hence the second integral vanishes. -What is left is Cauchy’s Formula. □
Ex. Integrate \( \frac{e^z}{z^2 - 1} \)
along the circle \(|z + 1| = 1\).

\[
\frac{1}{z^2 - 1} = \frac{1}{2} \left( \frac{1}{z - 1} - \frac{1}{z + 1} \right).
\]

\[
\oint_C \frac{e^z \, dz}{z^2 - 1} = \frac{1}{2} \oint_C \frac{e^z \, dz}{z - 1} - \frac{1}{2} \oint_C \frac{e^z \, dz}{z + 1}
= 0 - \frac{\pi i}{2} e^{-1} = -\frac{\pi i}{e}.
\]

\text{DERIVATIVES} \text{ An analytic function has derivatives of all orders. Recall that in the definition merely the first derivative is required. This can be read off from Cauchy's formula.}

\[
\frac{d}{dz} \oint_C \frac{f(\zeta) \, d\zeta}{\zeta - z}
= \frac{d}{dz} \oint_C \frac{f(\zeta) \, d\zeta}{(\zeta - z)^2}
= \frac{d^2}{dz^2} \oint_C \frac{f(\zeta) \, d\zeta}{(\zeta - z)^3}
\]

\text{...}
In general,

\[ \oint (z) = \frac{n!}{2\pi i} \oint \frac{f(\xi) d\xi}{(\xi - z)^{n+1}} \]

Ex.

\[ \oint \frac{\sin z \, dz}{(z - i)^3} = \frac{2\pi i}{1!} \left[ \frac{d^2 \sin z}{d z^2} \right]_{z = i} \]

\[ = -\pi i \sin(i) = \pi \sinh(1) \]

**Liouville's Theorem**

If \( f(z) \) is analytic and bounded in the whole complex plane, then \( f(z) \equiv \text{Constant} \).

**Proof:** By assumption \( |f(z)| \leq M \), where all \( z \). Fix a point \( z_0 \) and integrate along the circle \( |z - z_0| = R \) in the formula

\[ f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} \, dz. \]
\[
\left| \frac{1}{2\pi i} \frac{f(z)}{(z-z_0)^2} \right| = \frac{|f(z)|}{2\pi R} \leq \frac{M}{2\pi R^2}
\]

on the circle. By the ML-inequality,

\[
|f'(z_0)| \leq \frac{M}{2\pi R^2} \cdot 2\pi R = \frac{M}{R}.
\]

As \( R \to +\infty \) this estimate yields that \( f'(z_0) = 0 \). Since \( z_0 \) was an arbitrary point, this shows that \( f'(z) \equiv 0 \). Hence \( f(z) \equiv \text{Constant} \). Indeed,

\[
f(z) = f(0) + \int_{0}^{z} \frac{f'(z)}{z}dz = f(0) + 0.
\]

Liouville's Theorem can be used to prove the Fundamental Theorem of Algebra: Every polynomial

\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \quad (a_n \neq 0)
\]

has at least one zero. The reasoning goes as follows. If \( P(z) \neq 0 \) for all \( z \) (antithesis), then

\[
f(z) = \frac{1}{P(z)}
\]

is analytic in the whole complex plane (we never were to divide by 0). One can show that \( f(z) \) is bounded in the whole complex plane. By Liouville's Theorem \( f(z) \) is a constant. This is absurd. Thus there is a zero.
POWER SERIES

For a series
\[ \sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots \]
we form the partial sums
\[ S_1 = z_1, \quad S_2 = z_1 + z_2, \quad S_3 = z_1 + z_2 + z_3, \ldots, \]
\[ S_n = z_1 + z_2 + \cdots + z_n, \ldots \]
If there is a complex number \( S \) such that
\[ \lim_{n \to \infty} S_n = S, \]
then we say that the series \( \sum z_n \)
is \underline{convergent} and that \( S \) is its sum. This is denoted by
\[ \lim_{n \to \infty} \sum_{n=1}^{\infty} z_n = S. \]
In all other cases the series is called \underline{divergent}. For example, the harmonic series
\[ \sum \frac{1}{n} \]
is divergent. Indeed,
\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = + \infty \]

**Lemma** \[ \sum |z_n| < \infty \implies \sum z_n \text{ is absolutely convergent!} \]
THE GEOMETRIC SERIES $1 + z + z^2 + \cdots$

\[ 1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}, \quad z \neq 1 \]

$n$ terms are added up

$1 + z + z^2 + \cdots = \frac{1}{1 - z}, \text{ when } |z| < 1$

The geometric series diverges, when $|z| \geq 1$.

We say that $R = 1$ is the radius of convergence.

The following manipulations are valid, when $|z| < 1$. By differentiation

\[ \frac{1}{1 - z} = 1 + z + z^2 + z^3 + z^4 + z^5 + \cdots \]

\[ \frac{1!}{(1 - z)^2} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots \]

\[ \frac{2!}{(1 - z)^3} = 2 + 3 \cdot 2z + 4 \cdot 3z^2 + 5 \cdot 4z^3 + \cdots \]

\[ \frac{3!}{(1 - z)^4} = 3 \cdot 2 + 4 \cdot 3 \cdot 2z + 5 \cdot 4 \cdot 3z^2 + \cdots \]

\[ \vdots \]

and by integration

\[ \ln \frac{1}{1 - z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \]
Writing \( z = r e^{i \theta}, \ 0 \leq r < 1, \)

\[
1 + n e^{i \theta} + n^2 e^{2i \theta} + n^3 e^{3i \theta} + \ldots = \frac{1}{1 - re^{i \theta}}
\]

and taking real and imaginary parts, we obtain the interesting sums:

\[
1 + n \cos \theta + n^2 \cos 2\theta + n^3 \cos 3\theta + \ldots = \frac{1 - n \cos \theta}{1 + n^2 - 2n \cos \theta}
\]

\[
r \sin \theta + n^2 \sin 2\theta + n^3 \sin 3\theta + \ldots = \frac{n \sin \theta}{1 + n^2 - 2n \cos \theta}
\]

both valid, when \( 0 \leq n < 1. \)

---

**POWER SERIES**

\[
S(z) = \sum_{n=0}^{\infty} a_n (z - z_0) = a_0 + a_1 (z - z_0) + \]

\[
+ a_2 (z - z_0)^2 + \ldots + a_{n-1} (z - z_0)^{n-1} + \ldots
\]

Often, the "center" \( z_0 = 0. \)

**ABEL'S THEOREM.** If the power series converges at the point \( z = z_1 \neq z_0, \) then it converges (even absolutely) at all points \( z \) for which \( |z - z_0| < |z_1 - z_0|. \)

If the power series diverges at the point...
\[ Z_2, \text{ then it diverges at all points } Z \text{ for which } |Z - Z_0| > |Z_1 - Z_0|. \]

Convergence inside of the small circle.

Divergence outside of the large circle.

Abel's theorem implies the existence of a so called RADIUS OF CONVERGENCE.

To each power series \[ \sum a_n (z-z_0)^n \] there corresponds a radius of convergence \( R, 0 \leq R \leq +\infty \) such that the series converges (even absolutely), when \( |z - z_0| < R \) and diverges when \( |z - z_0| > R \). (If \( R = 0 \), the series diverges, when \( z \neq z_0 \). If \( R = \infty \), the series converges for all \( z \).)

Please, notice that nothing is said about points on the circle of convergence \( |z - z_0| = R \).

\[
\begin{align*}
\sum n! z^n & \quad \frac{\sum z^n}{R = 0} & \quad \sum z^n & \quad \frac{\sum z^n}{R = 1} & \quad \sum \frac{z^n}{n!} & \quad \frac{R = +\infty}{\text{Geometric series}} & \quad \frac{\text{Always convergent}}{e^z}
\end{align*}
\]
\[ R = 1 \text{ if all } \frac{2}{n!} \text{ for all } \frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \rightarrow 0 \text{ if } R = \frac{1}{0} = +\infty. \]

\[ \sum n^{-1338}z^n, \sum n^{-1}z^n, \sum z^n, \sum n^{1338}z^n \]

\[ R = 1 \text{ for all } \]

**Strictly inside the circle of convergence we can differentiate and integrate a power series termwise.**

**Example** \( S(z) = \sum n z^n = ? \) \( R = 1 \)

\[ S(z) = 1 \cdot z + 2 z^2 + 3 z^3 + \cdots \quad |z| < 1 \]

\[
\frac{S(z)}{z} = 1 + 2 z + 3 z^2 + \cdots \\
= \frac{d}{dz} \left\{ z + z^2 + z^3 + \cdots \right\} = \frac{d}{dz} \left( \frac{z}{1-z} \right) \\
= \frac{1}{(1-z)^2} ; \quad S(z) = \frac{z}{(1-z)^2} \quad S(0) = 0 \text{ as it should be.} \\
\]

\[ 1z + 2 z^2 + 3 z^3 + \cdots = \frac{z}{(1-z)^2} , \text{ when } |z| < 1 \]
The function
\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \]
is analytic, when \( |z - z_0| < R = \text{radius of convergence} \). Moreover
\[ a_n = \frac{f^{(n)}(z_0)}{n!} \]

Actually, all analytic functions are power series!

More precisely:

**THEOREM** Let \( f(z) \) be analytic in the domain \( \Omega \) and let \( z_0 \) be any interior point in \( \Omega \). Then the expansion
\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \]
\[ a_n = \frac{f^{(n)}(z_0)}{n!} \]
is valid in the largest disc contained in \( \Omega \) with center \( z_0 \).

\[ |z - z_0| < R = \text{the distance from } z_0 \text{ to the boundary of the domain } \Omega. \]

**Proof:** Start with Cauchy's Formula
\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \left\{ 1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots \right\}
\]

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0} \, d\zeta + \frac{z - z_0}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^2} \, d\zeta + \frac{(z - z_0)^2}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^3} \, d\zeta + \cdots
\]

\[
= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!} (z - z_0)^2 + \cdots
\]

Recall the formula

\[
(f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.
\]

The expansion is called the TAYLOR (MACLAURIN) series.
\[ e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \]
\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \]
\[ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \]
\[ \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \]
\[ \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \]
\[ \tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \frac{62z^9}{2835} + \cdots \]
when \(|z| < \frac{\pi}{2}\)
\[ \ln (1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad |z| < 1 \quad (\ln 1 = 0) \]
\[ \arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \quad |z| < 1 \]
\[ \log \left( \frac{1+z}{1-z} \right) = 2 \left( z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots \right) \quad |z| < 1 \]
\[ (1 + z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \frac{p(p-1)(p-2)}{3!} z^3 + \cdots \]

Converges at least when \(|z| < 1\).
\[ (a_n z^n) + (a_n b_n z^n) + \ldots + (a_n b_n z^n) + \ldots = a_0 + b_0 + (a_0 b_0 + a_0 b_0) z + a_0 b_2 z^2 + a_0 b_2 z^3 + \ldots + a_0 b_2 z^3 + \ldots + a_0 b_3 z^3 + \ldots \]

ADD UP

\[ a_0 + b_0 z + b_2 z^2 + b_3 z^3 + \ldots + a_0 + b_0 z + a_0 b_2 z^2 + a_0 b_2 z^3 + \ldots + a_0 + b_0 z + a_0 b_2 z^2 + a_0 b_2 z^3 + \ldots \]

Cauchy's rule:

When \( |z| < \min \{R_0, R_1 \} \), the series has a radius of convergence.

\[ \sum a_n z^n + \sum b_n z^n = \sum (a_n + b_n) z^n \]
Ex. \[ \frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x} = (1 + z + z^2 + z^3 + \cdots) \cdot (1 + z + z^2 + z^3 + \cdots) = 1 \cdot 1 + (1 \cdot 1 + 1 \cdot 1) z + (1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) z^2 + \cdots = 1 + 2z + 3z^2 + 4z^3 + \cdots, \quad |z| < 1. \]

Ex: \[ \tan z = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < \frac{\pi}{2} \]

Use the formula \[ \frac{d\tan z}{dz} = 1 + \tan^2 z \]

to calculate the coefficients:
\[ a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots = 1 + a_1 z^2 + 2a_1 a_2 z^3 + (2a_1 a_3 + a_2^2) z^4 + \cdots \]

Comparing coefficients:
\[ a_1 = 1, \quad 2a_2 = 0, \quad 3a_3 = a_1^2, \quad 4a_4 = 2a_1 a_2, \quad 5a_5 = 2a_1 a_3 + a_2^2. \]
\[ a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{1}{3}, \quad a_4 = 0, \quad a_5 = \frac{2}{15}. \]

\[ \tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots \]

Accidentally, we have solved the differential equation \[ y' = 1 + y^2, \quad y(0) = 0. \]
THE LAURENT SERIES

The expansion

\[
\frac{\sin z}{z^5} = z^{-4} - \frac{1}{6} z^{-3} + \frac{1}{120} - \frac{1}{5040} z^2 + \cdots,
\]

0 \leq |z| < \infty, is not a Maclaurin (= Taylor, \(z_0 = 0\)) series, since it contains negative powers. It is a LAURENT series.

THEOREM Suppose that \( f(z) \) is analytic in the annulus \( R_1 < |z-z_0| < R_2 \). Then \( f(z) \) can be represented by the Laurent series

\[
f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + \]

\[+ a_1 (z-z_0) + a_2 (z-z_0)^2 + \cdots\]

when \( R_1 < |z-z_0| < R_2 \).

Moreover,

\[
a_{-1} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz
\]

where \( C \) is a closed curve in the annulus.

REMARK: The case \( R_1 = 0 \) is the most significant. Then \( f(z) \) is analytic in the punctured disc \( 0 < |z-z_0| < R_2 \).
Ex. \[ \frac{1}{z^2-1} = \frac{1/2}{z-1} - \frac{1}{4} + \frac{1}{8} \frac{(z-1)}{z-1} - \frac{1}{16} \frac{(z-1)^3}{z-1} + \cdots, \quad \text{when} \quad 0 < |z-1| < 2. \]

\[ \frac{1}{z^2-1} = \frac{1}{z-1} \left( \frac{1}{1+(z-1)} \right) = \frac{1}{z(z-1)} \left( 1 - \left( \frac{z-1}{2} \right) \right) \]

\[ = \frac{1}{2(z-1)} \left\{ 1 - \frac{z-1}{2} + \left( \frac{z-1}{2} \right)^2 - \left( \frac{z-1}{2} \right)^3 + \cdots \right\} \]

Ex.: \[ e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! \cdot z^2} + \frac{1}{3! \cdot z^3} + \cdots, \quad z \neq 0. \]

We say that \( f(z) \) has a Pole of the \( n \)th order at \( z_0 \), if the Laurent expansion has the form

\[ f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots, \]

\( a_{-n} \neq 0 \), when \( 0 < |z-z_0| < \text{some radius} \). If the Laurent expansion has infinitely many terms with negative powers, we talk about an essential singularity. For example, the origin is an essential singularity for \( e^{1/z} \).
CALCULUS OF RESIDUES

**Basic Result**

\[ \oint_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} \{ f(z) \} \]

Here \( f(z) \) is analytic inside and \( C \), except at the points \( z_1, z_2, \ldots, z_n \).

Let us begin with a simpler situation. Suppose that we have the Laurent expansion:

\[ f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \ldots \]

when \( 0 < |z-z_0| < R \) (near \( z_0 \)).

Integrating termwise we get:

\[ \oint_C f(z) \, dz = 2\pi i a_{-1}, \]

Since

\[ \oint_C \frac{dz}{(z-z_0)^k} = 0, \quad k = 0, -1, \pm 2, \ldots \]

and

\[ \oint_C \frac{dz}{z-z_0} = 2\pi i. \]
We say that the coefficient \( a_{-1} \) is the residue of \( f(z) \) at the point \( z_0 \):

\[
a_{-1} = \text{Res} \frac{f(z)}{z - z_0}
\]

Returning to the general case with several singularities, we have

\[
\oint_{C} f(z) \, dz = 2\pi i \sum_{k} \text{Res} \frac{f(z)}{z - z_k}
\]

where \( C_k \) is a small circle around the singular point \( z_k \). The circles are not allowed to overlap.

\[
\oint_{C} f(z) \, dz = \oint_{C_1} f(z) \, dz + \cdots + \oint_{C_n} f(z) \, dz
\]

\[
= 2\pi i \left( \text{Res} \frac{f(z)}{z - z_1} + \cdots + \text{Res} \frac{f(z)}{z - z_n} \right)
\]

This is the desired formula for evaluating the integral in terms of residues.

\[
163 = 13^2, \quad 261 = 31^2.
\]
EXAMPLE

\[ \oint_C \frac{dz}{1-z^2} = 2\pi i \left\{ \text{Res}_{z=1} \frac{1}{1-z^2} + \text{Res}_{z=-1} \frac{1}{1-z^2} \right\} \]

\[ z = 1 \]

\[ \frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)} \]

\[ = \frac{1}{(1-z)(2+(z-1))} = \frac{-1/2}{(z-1)(1+\frac{z-1}{2})} = \]

\[ 0 < |z-1| < 2 \]

\[ = -\frac{1}{2(z-1)} \left\{ 1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \cdots \right\} \]

\[ = \frac{-1/2}{z-1} + \frac{1}{4} - \frac{1}{8} (z-1) + \cdots ; \quad \text{Res}_{z=1} \frac{1}{1-z^2} = -\frac{1}{2} \]

\[ z = -1 \]

In a similar manner, when \( 0 < |z+1| < 2 \),

\[ \frac{1}{1-z^2} = \frac{1/2}{z+1} - \frac{1}{4} + \frac{1}{8} (z+1) + \cdots ; \quad \text{Res}_{z=-1} \frac{1}{1-z^2} = \frac{1}{2} \]

Answer: \( \oint_C \frac{dz}{1-z^2} = 2\pi i \left\{ \frac{1}{2} - \frac{1}{2} \right\} = 0 \).

At a SIMPLE pole \( z_0 \),

\[ \text{Res} \left\{ f(z) \right\} = \lim_{z \to z_0} [(z-z_0) f(z)] \]
At a pole of the second order
\[ \text{Res} \ f(z) = \lim_{z \to z_0} \frac{d}{dz} \left[ \frac{(z-z_0)^2 f(z)}{z} \right] \]

**INTEGRALS**
\[
\int_0^{2\pi} R(\sin \theta, \cos \theta) \, d\theta = \frac{\pi}{i \pi} \left\{ \begin{array}{c}
\sin \theta = \frac{e^{i \theta} - e^{-i \theta}}{2i} = \frac{1}{2i} (z - \frac{1}{z}) \\
\cos \theta = \frac{e^{i \theta} + e^{-i \theta}}{2} = \frac{1}{2} (z + \frac{1}{z})
\end{array} \right.
\]

This transforms the integral to a curve integral along the unit circle \(|z| = 1\).

**Ex.**
\[
I = \int_0^{2\pi} \frac{d\theta}{5 + 3 \cos \theta} = \int_0^{2\pi} \frac{\frac{dz}{i \pi}}{5 + \frac{3}{2} (z + \frac{1}{z})}
\]
\[
= \cdots = - \frac{2i}{3} \int_{|z|=1} \frac{dz}{(z + \frac{1}{3})(z + 3)}
\]
\[
= - \frac{2i}{3} \cdot 2\pi i \cdot \text{Res} \left. \frac{1}{(z + \frac{1}{3})(z + 3)} \right|_{z = -\frac{1}{3}} = \frac{4\pi i}{3} \cdot \frac{1}{-\frac{1}{3} + 3} = \frac{11\pi}{2}
\]

Notice that the pole at \(z = -3\) is not included, since it is outside the contour \(|z| = 1\).
Inequalities like \[ \int_{-\infty}^{\infty} f(x) \, dx \]

Try to integrate over a large half-circle and let the radius go to \( +\infty \).

Ex. \[ I = \int_{\infty}^{-\infty} \frac{dx}{x^2 + 2x + 2} \]

\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} + 2 \int_{0}^{\pi} \frac{i \Re e^{i\theta} \, d\theta}{R^2 e^{2i\theta} + 2 i \Re e^{i\theta} + 2} = \oint \frac{dz}{z^2 + 2z + 2}
\]

\[
= 2\pi i \left[ \frac{\Re}{z^2 + 2z + 2} \right]_{z = -1 + i} = \frac{2\pi i}{2i} = \pi
\]

\[ \text{Poles: } -1 + i \text{ in the upper half-plane.} \]

On the half-circle \( z = R e^{i\theta} \)

\[ |z^2 + 2z + 2| \geq |z^2| - |2z + 2| \geq R^2 - 2R - 2 \]

By the ML-inequality

\[ |\oint_{\gamma} dz| \leq \frac{\pi R}{R^2 - 2R - 2} \rightarrow 0 \quad \text{as } R \to +\infty \]

Result:

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi. \]
Remark: If $P(z)$ and $Q(z)$ are polynomials, such that 1°) $Q(x) \neq 0$ (no zeros on the real axis) and 2°) degree $Q \geq 1 + \text{degree } P$, then the integral over the half-circle vanishes as its radius goes to $\infty$. We are left with the formula

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx = 2\pi i \sum_{\gamma > 0} \text{Res} \left( \frac{P(z)}{Q(z)} \right)$$

where the degree of the denominator is at least two units higher than the degree of the numerator.

where the degree of the denominator $\geq 2$ at least two units higher than the degree of the numerator.

The sum of the residues in the upper half-plane

$$\int_{0}^{\infty} \frac{1 + x^3}{1 + x^4} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1 + x^3}{1 + x^4} \, dx$$

$$\int_{-\infty}^{\infty} \frac{1 + x^3}{1 + x^4} \, dx = 2\pi i \left\{ \text{Res} \left( \frac{1 + z^3}{1 + z^4} \right)_{z = \frac{1 + \sqrt{2}i}{1+\sqrt{2}}} + \text{Res} \left( \frac{1 + z^3}{1 + z^4} \right)_{z = \frac{-1 + \sqrt{2}i}{1+\sqrt{2}}} \right\}$$

There are four simple poles, viz. $\frac{1 \pm \sqrt{2}i}{1+\sqrt{2}}$ (the roots of $1 + z^4 = 0$), but only two of them are in the upper half-plane. Hence only the residues at $\pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$ are included in the calculation. Please, calculate the residues needed yourself!

$$1233 = 12 \cdot 33^2, \quad 8833 = 88^2 + 33^2$$
Ex. \( \int_{0}^{\infty} \frac{\ln x}{x} \, dx = \frac{\pi}{2} \)

The function \( f(z) = \frac{\ln z}{z} \) will not work here! Consider instead

\[
\frac{e^{iz}}{z} = \frac{\cos z + i \sin z}{z} \quad (z \neq 0)
\]

and integrate along the contour:

![Contour Diagram]

The half-circles have radii \( \varepsilon \) and \( R \), 
\( 0 < \varepsilon < R \)

\[
0 = \oint \frac{e^{iz}}{z} \, dz = \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} \, dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} \, dx + \int_{C_{R}} \frac{e^{iz}}{z} \, dz
\]

\[
+ \int_{C_{\varepsilon}} \frac{e^{iz}}{z} \, dz
\]

On the large half-circle

\[
|z| = R, \quad \left| \frac{e^{iz}}{z} \right| = \frac{e^{-R \sin \theta}}{R}, \quad \frac{dz}{z} = i \, d\theta
\]

\[
\left| \int_{C_{R}} \frac{e^{iz}}{z} \, dz \right| \leq \int_{0}^{\pi} e^{-R \sin \theta} \, d\theta \quad \xrightarrow{R \to \infty} 0
\]

\[
\int_{C_{\varepsilon}} \frac{e^{iz}}{z} \, dz = 0
\]
Thus

\[ 0 = \int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x} \, dx + \int_{-\epsilon}^{\epsilon} \frac{e^{ix}}{x} \, dx + \int_{\epsilon}^{\infty} \frac{e^{iz}}{z} \, dz \]

\[ z = \epsilon e^{i\theta} \]

\[ \pi \geq \theta \geq 0 \]

\[ \rightarrow -\pi i \]

By symmetry, \( \left( \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\cos x}{x} \, dx = 0 \).

The integral along the small half-circle is

\[ \int_{\pi i}^{\epsilon i} \frac{e^{iz}}{z} \, dz = \int_{\pi i}^{\epsilon i} \frac{dz}{z} + \int_{\pi i}^{\epsilon i} \frac{iz}{z} \, dz + \int_{\pi i}^{\epsilon i} \frac{z^2}{2!} \, dz + \ldots \]

\[ = -\pi i \]

\[ \rightarrow -\pi i \quad (\text{clockwise}) \]

The result is

\[ \int_{-\infty}^{\infty} \frac{\min x}{x} \, dx = \pi i. \]

Remark:

\[ \int_{-\infty}^{\infty} \left| \frac{\min x}{x} \right| \, dx = +\infty \]

That is, how it is!
THE FOURIER TRANSFORM

\[
\begin{cases}
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \\
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{+i\omega x} \, d\omega
\end{cases}
\]

Notation: \( \mathcal{F}(f(x)) = \hat{f}(\omega) \). Ex. \( \mathcal{F}(e^{-ax^2}) = \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2a}} \).

\( \mathcal{F}(af(x) + bg(x)) = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)) \) \quad \text{LINEARITY}

Convolution: \((f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \, dy \) \quad \text{(DEFINITION)}

\[
\mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g)
\]

Example  \( \mathcal{F}\left(\frac{1}{1+x^2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+x^2} \, dx \)

\[= \cdots = \sqrt{\frac{\pi}{a}} e^{-|\omega|} \] Integrate along a large half-circle (keep \( \omega \cdot \text{Im} z \leq 0 \)); residues. \( \begin{array}{c}
\text{Residues.} \quad \text{C}
\end{array} \)

The contour, if \( \omega > 0 \).