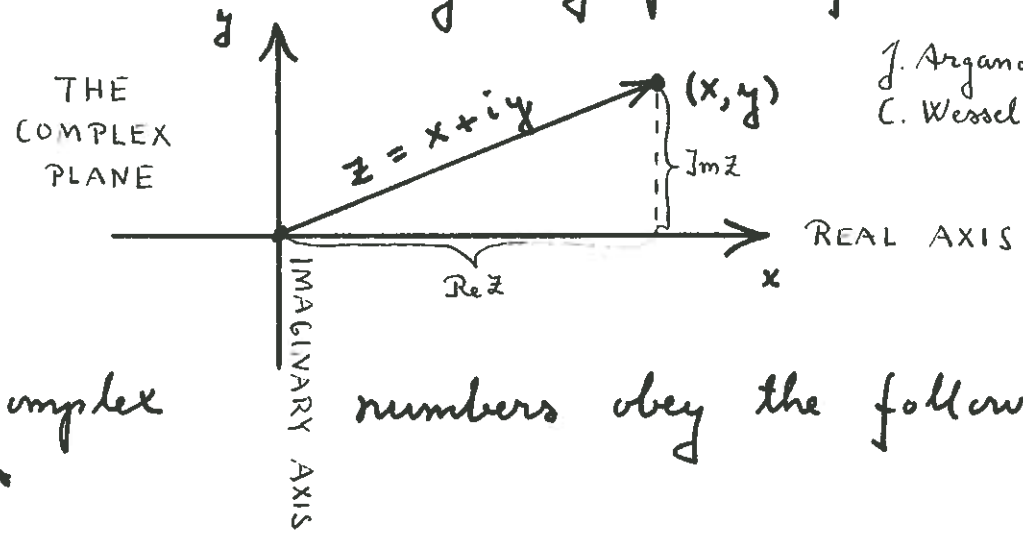
SYNOPSIS

COMPLEX NUMBERS

$\boxed{z = x + iy}$ (x and y are real numbers)

$x = \mathrm{Re}\, z$ = the <u>real</u> <u>part</u> of z

$y = \mathrm{Im}\, z$ = the <u>imaginary</u> <u>part</u> of z



J. Argand 1806
C. Wessel 1797

The complex numbers obey the following axioms

AXIOMS

I $\quad z_1 = x_1 + iy_1 = x_2 + iy_2 = z_2 \iff x_1 = x_2$ & $y_1 = y_2$ IDENTITY

II $\quad z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)$ ADDITION

III $\quad z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$ MULTIPLICATION

$$= x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2)$$

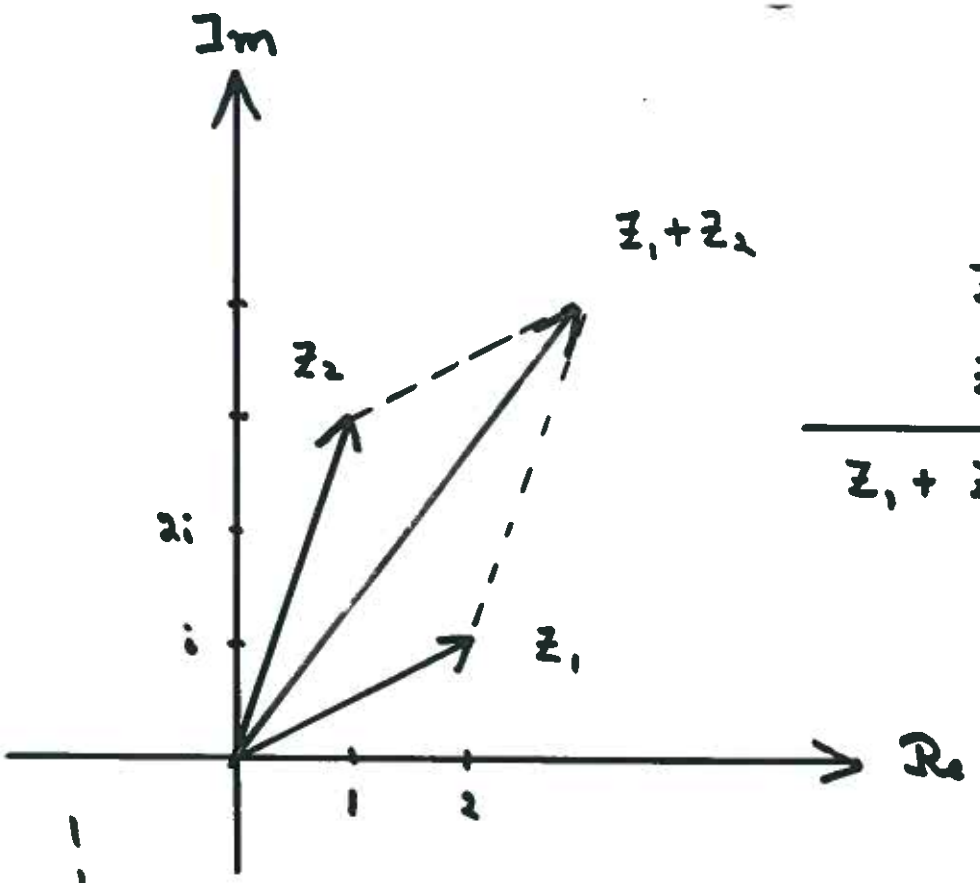
We write $x = x + i0$, $iy = 0 + iy$ (purely imaginary numbers), $0 = 0 + i0$. Also $x + iy = x + yi$.

$\boxed{i^2 = -1}$

<u>Examples</u> $1 + 3i \neq 1{,}002 + 3i$

$(1 + 3i) + (-7 + i) = -6 + 4i$

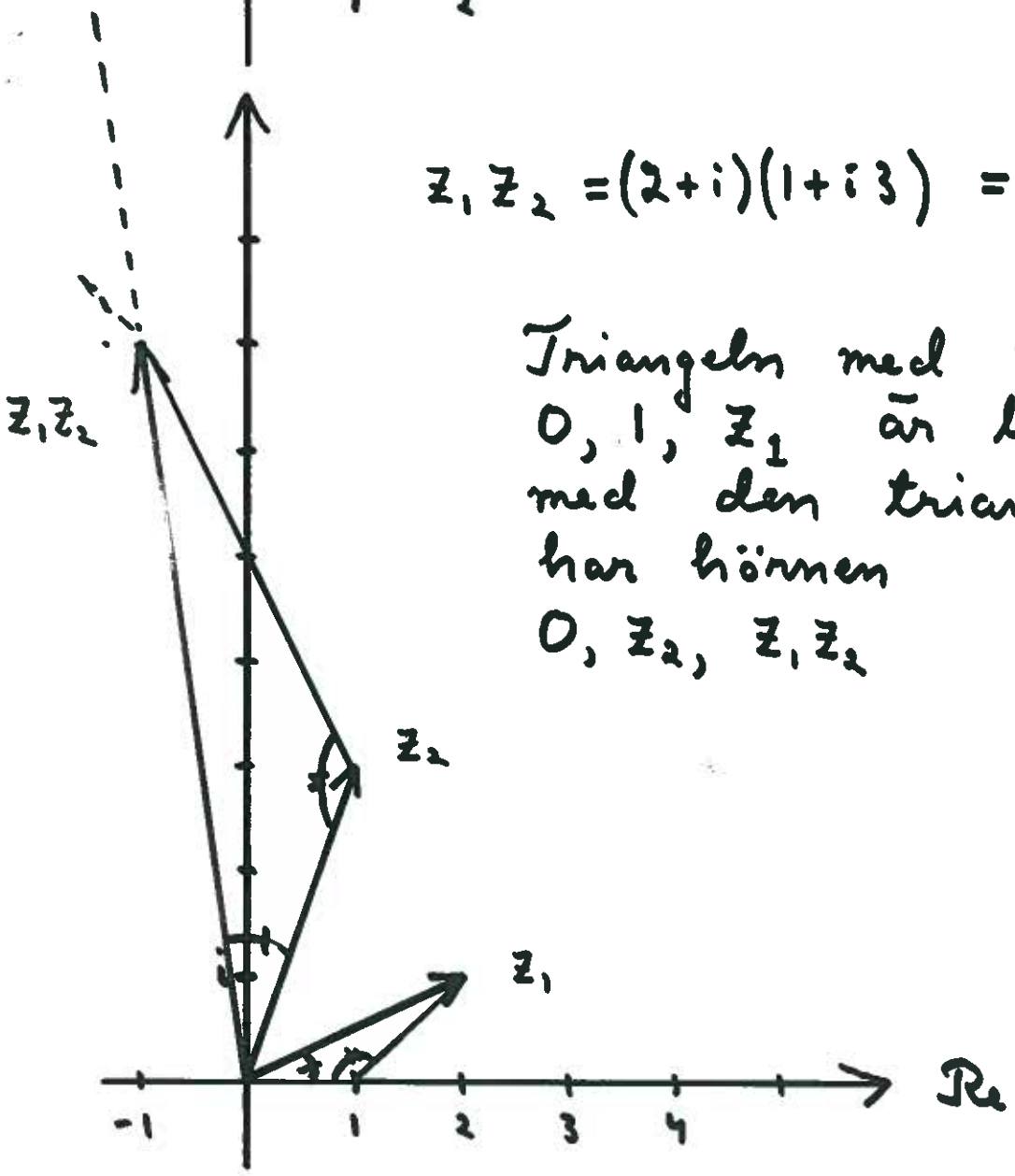
$(2 - i)(1 + 3i) = 5 + 5i = 5(1 + i)$



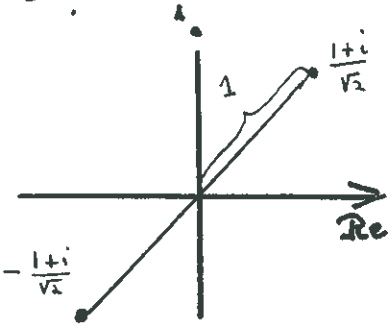
$$z_1 = 2+i$$
$$z_2 = 1+i3$$
$$z_1+z_2 = 3+i4$$

$$z_1 z_2 = (2+i)(1+i3) = -1+i7$$

Triangeln med hörnen $0, 1, z_1$ är likformig med den triangel som har hörnen $0, z_2, z_1 z_2$



Example Solve the equation $z^2 = i$.



$(x+iy)^2 = i$, $\quad x^2 - y^2 + 2xyi = i$,

$$\begin{cases} x^2 - y^2 = 0 \iff x = \pm y \\ 2xy = 1 \end{cases}$$

We get $x, y = \pm 1/\sqrt{2}$. Notice that x and y must have the same sign, since $2xy = 1 > 0$. We obtain

$$z = \pm \frac{1+i}{\sqrt{2}}, \quad \text{or} \quad \text{"}\sqrt{i}\text{"} = \pm \frac{1+i}{\sqrt{2}}$$

Remark When solving $z^2 = a + ib$ it is worth noticing that

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = a^2 + b^2.$$

RULES:

$$\boxed{\begin{array}{lll} z_1 + z_2 = z_2 + z_1 & \text{commutation} & z_1 z_2 = z_2 z_1 \\ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 & \text{association} & z_1(z_2 z_3) = (z_1 z_2) z_3 \\ & \text{distribution} & z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \end{array}}$$

Example $z^5 + z + 1 = (z^2 + z + 1)(z^3 - z^2 + 1)$

The multiplication can be performed as for real numbers, for instance,

$$(100 + 10 + 1)(1000 - 100 + 1) = 100000 + 10 + 1.$$

$$(b-a)(b+a) = b^2 - a^2.$$

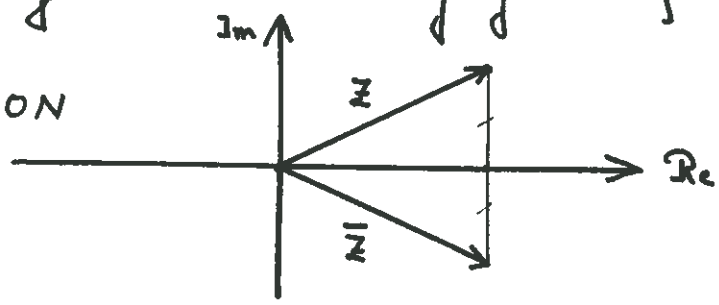
Next page!

$$z + 0 = z, \quad 1z = z, \quad z + (-z) = 0, \quad z z^{-1} = 1.$$

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} \quad (z \neq 1)$$

$\bar{z} = x - iy$ is the conjugate of $z = x + iy$

CONJUGATION



Geometrically, conjugation is a reflexion in the real axis.

$$\overline{(\bar{z})} = z, \qquad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \qquad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \qquad z\bar{z} = x^2 + y^2$$

Notice that from $z = x + iy$, $\bar{z} = x - iy$ we get

$$x = \operatorname{Re} z = \frac{z + \bar{z}}{2}, \qquad y = \operatorname{Im} z = \frac{z - \bar{z}}{2i},$$

$$\boxed{z\bar{z} = x^2 + y^2}$$

$|z|^2 = z\bar{z}$

Defined later

<u>Example</u> $\overline{3 + 4i} = 3 - 4i$, $\overline{(1+i)^2} = (1-i)^2 = -2i$.

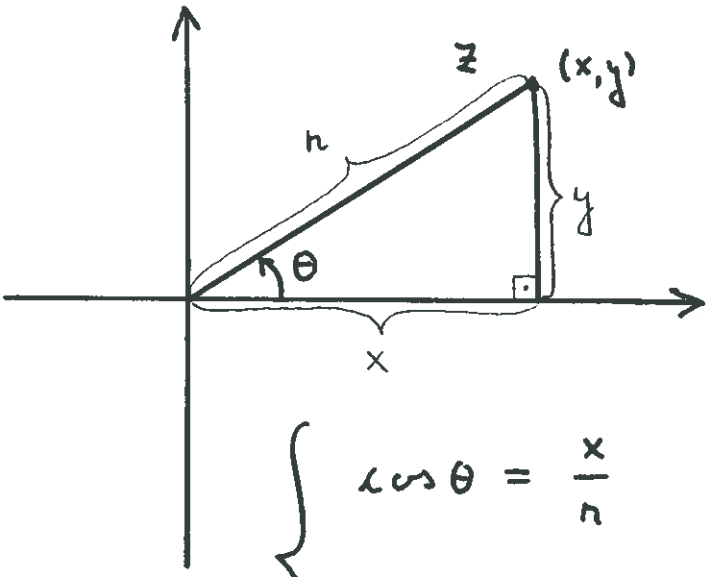
DIVISION $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{x - iy}{x^2 + y^2} \qquad (z \neq 0)$

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \cdots \qquad (z_2 \neq 0)$$

$$\frac{3+i}{2-i} = \frac{(3+i)(2+i)}{(2-i)(2+i)} = \frac{5+5i}{4+1} = 1 + i$$

NOTATION $z^{-1} = \dfrac{1}{z}$, $z \neq 0$. $\boxed{z z^{-1} = 1}$

MODULUS (= absolute value) and ARGUMENT



$$z = x + iy$$

$$r = +\sqrt{x^2 + y^2}$$

$$= \sqrt{z\bar{z}}$$

$$\begin{cases} \cos\theta = \dfrac{x}{r} \\ \sin\theta = \dfrac{y}{r} \end{cases}$$

$r = |z|$ = the <u>modulus</u> of z

$\theta = \arg z$ = the <u>argument</u> of z

$$\boxed{z = x + iy = r(\cos\theta + i\sin\theta)}$$

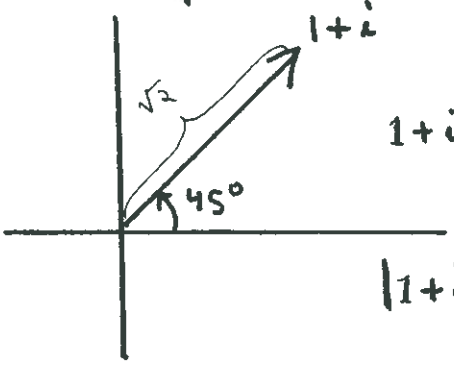
The argument is manyvalued: $\theta + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ will do as well.

LEMMA <u>Suppose that</u> $r_1 > 0$ <u>and</u> $r_2 > 0$. <u>Then</u>

$$r_1(\cos\theta_1 + i\sin\theta_1) = r_2(\cos\theta_2 + i\sin\theta_2) \iff$$

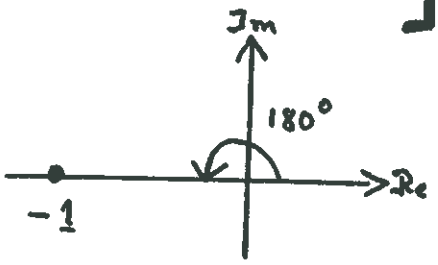
$$r_1 = r_2 \ \& \ \theta_1 = \theta_2 + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots$$

<u>Example</u>



$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$|1+i| = \sqrt{2}$$



$$-1 = 1(\cos\pi + i\sin\pi)$$

$$|-1| = 1$$

LEMMA $\quad |z_1 z_2| = |z_1||z_2|$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad (\text{modulo } 2\pi)$$

<u>Proof:</u> $\quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\qquad\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\underbrace{(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)}_{\sin(\theta_1+\theta_2)}\right]$$

$$= r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right].$$

$\square$

In particular

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta \qquad \theta+\theta = 2\theta$$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \qquad 2\theta+\theta = 3\theta$$

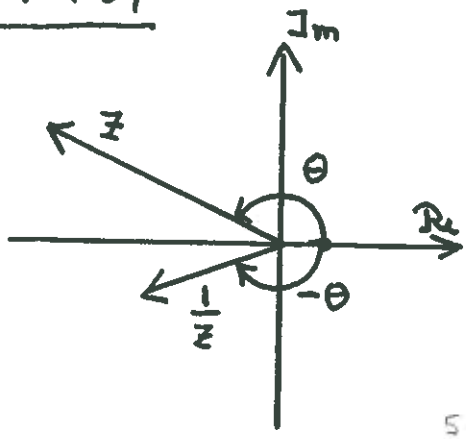
and so on.

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

de Moivre's formula. $n = 0, \pm 1, \pm 2, \ldots$

$$\frac{1}{z} = \frac{1}{r(\cos\theta_1 + i\sin\theta_1)} = \frac{\cos\theta_1 - i\sin\theta_1}{r}$$

$$= \frac{\cos(-\theta_1) + i\sin(-\theta_1)}{r}.$$



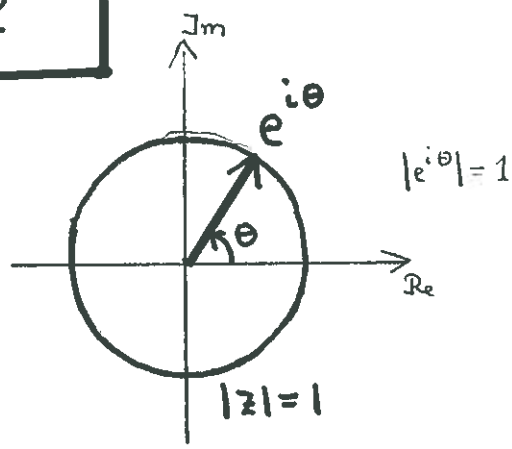
We have $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $\arg\left(\frac{z_1}{z_2}\right) =$

$= \arg z_1 - \arg z_2$, when $z_2 \neq 0$.

$i = e^{i\frac{\pi}{2}}$

EULER'S FORMULA

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$



$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

EULER

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

In particular,

$$e^{i\pi} = -1 \quad !$$

As we will learn later, the meaning of the formula above is that

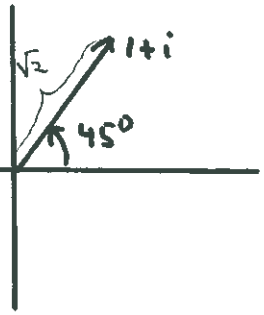
$$1 + i\pi + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \cdots = -1 \quad \text{or} \quad \lim_{n\to\infty}\left(1 + \frac{i\pi}{n}\right)^n = -1.$$

Now de Moivre's formula can be written as

$$\boxed{\left(e^{i\theta}\right)^n = e^{in\theta}}$$ de MOIVRE

$$\underline{z = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

<u>Example:</u> $1+i = \sqrt{2}\, e^{i\pi/4}$



$$e^{i(\theta+2k\pi)} = e^{i\theta} \quad ! \quad \text{Period } 2k\pi i$$

EXTRACTION OF ROOTS

DEF.: $$z^n = w \overset{\text{DEFINITION}}{\Longleftrightarrow} z = \sqrt[n]{w}$$

For example, $\sqrt{i} = \sqrt[2]{i} = \pm\dfrac{1+i}{\sqrt{2}}$; $\sqrt{1} = \pm 1$; $\sqrt[4]{1} = 1, -1, i, -i$.

$$w = \rho e^{i\psi} \quad (\rho > 0), \quad z = r e^{i\theta},$$

The problem is to express r and θ in terms of ρ and ψ: $r = ?$, $\theta = ?$

$$(r e^{i\theta})^n = \rho e^{i\psi},$$

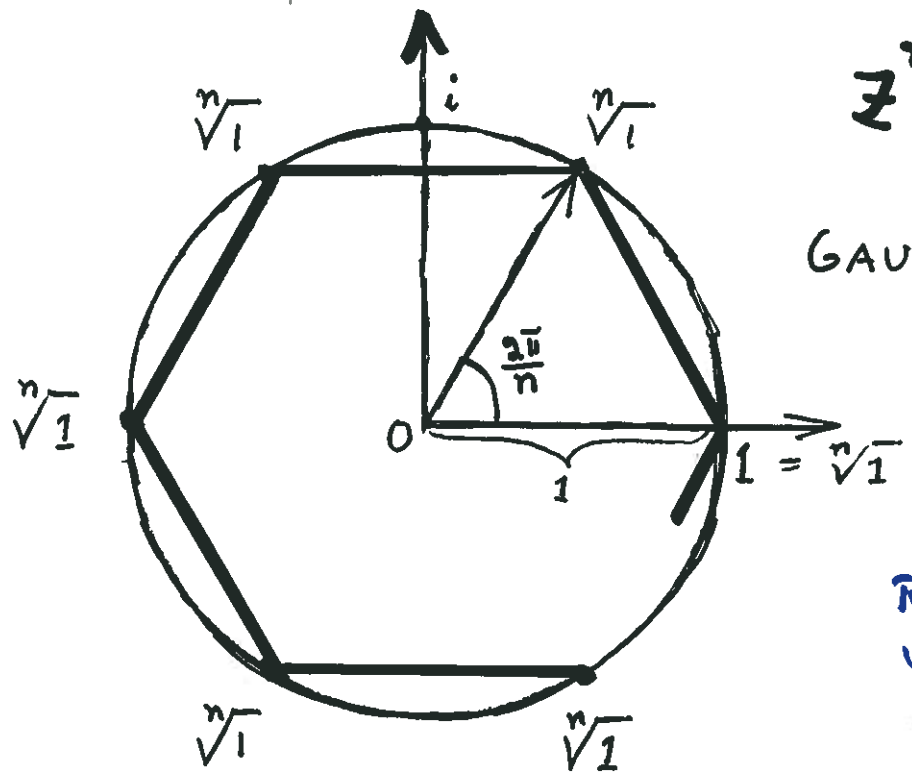
$$r^n e^{in\theta} = \rho e^{i\psi},$$

$$r^n = \rho \quad \& \quad n\theta = \psi + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

$$\begin{cases} r = \sqrt[n]{\rho} \quad (\text{the } \underline{\text{real}} \text{ root, a positive number}), \\ \theta = \dfrac{\psi}{n} + \dfrac{2k\pi}{n}. \end{cases}$$

$$\boxed{\sqrt[n]{w} = \sqrt[n]{\rho}\, e^{i\left(\frac{\psi}{n} + \frac{2k\pi}{n}\right)} \quad (k = 0, \pm 1, \pm 2, \ldots)}$$

The nth root has exactly n different values, since the exponential has the period $2\pi i$. The choice $k = 0, 1, 2, 3, \ldots, n-1$ for instance, yields all the roots.



$$z^n = 1$$

GAUSS 1777 – 1855

ROOTS OF UNITY

Rötterna

$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \qquad (k = 0, 1, 2, \ldots, n$$

till ekvationen $z^n = 1$ bildar <u>en</u> <u>reguljär n-hörning</u>. Enligt Gauss kan denna polygon konstrueras med passare och linjal om och endast om

$$n = 2^m p_1 p_2 \cdots p_k, \qquad \underline{\text{EX.}}\ 4 \cdot 5 \cdot 257$$

där $m = 0, 1, 2, \ldots$ och p:na är <u>olika</u> <u>primtal</u> av formen

$$p = 2^{(2^j)} + 1 \qquad (\text{Fermats primtal}).$$

FERMATS PRIMTAL: $\underbrace{3, 5, 17, 257, 65537,}_{\text{Alla kända tills vidare!}}$

$$2^{2^5} + 1 = 641 \cdot 6700417 \qquad (\text{EULER } 1732)$$

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 51, ...

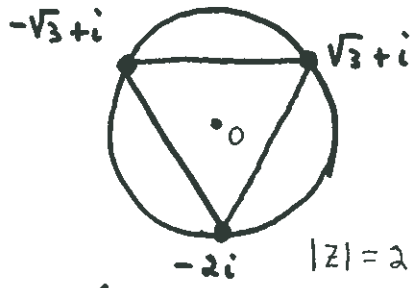
<u>Exempel</u> $\sqrt[3]{8i} = ?$

$$z^3 = 8i, \quad \left(re^{i\theta}\right)^3 = 8e^{i\frac{\pi}{2}}, \quad r^3 e^{3i\theta} = 8e^{i\frac{\pi}{2}},$$

$$\begin{cases} r^3 = 8, & r = 2 \\ 3\theta = \frac{\pi}{2} + 2k\pi \end{cases}$$

Hence $r = 2$ and $\theta = \frac{\pi}{6} + \frac{2k\pi}{3}$, $k = 0, \pm 1, \pm 2, \ldots$

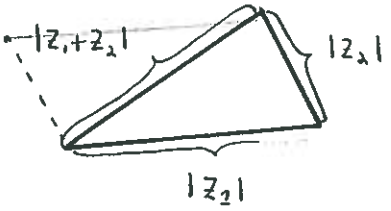
$$z = \begin{cases} 2e^{i\pi/6} = \sqrt{3} + i & k = 0 \\ 2e^{i5\pi/6} = -\sqrt{3} + i & k = 1 \\ 2e^{i3\pi/2} = -2i & k = 2 \end{cases}$$



The 3nd root has exactly 3 different values.

THE TRIANGLE INEQUALITIES

$$\pm\left(|z_1| - |z_2|\right) \le |z_1 + z_2| \le |z_1| + |z_2|$$



THE BINOMIAL FORMULA

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 +$$

$$+ \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n$$

The coefficient in front of $a^{n-k}b^k$ is

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ and can be found

from PASCAL's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6

POLYNOMIALS, ALGEBRAIC EQUATIONS, THE FUNDAMENTAL THEOREM OF ALGEBRA

$$P(z) = \overbrace{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}^{\text{polynomial}}, \quad a_n \neq 0$$

†) $$\underline{P(z) - P(z_1)} = a_1(z - z_1) + a_2(z^2 - z_1^2) + \cdots + a_n(z^n - z_1^n)$$

$$= (z - z_1)\Big\{a_1 + a_2(z + z_1) + a_3(z^2 + z z_1 + z_1^2) + \cdots + a_n(z^{n-1} + z^{n-2} z_1 + \cdots + z z_1^{n-2} + z_1^{n-1})\Big\}$$

$= \underline{(z - z_1) \cdot \{\text{"a polynomial"}\}}$. This means that if $P(z_1) = 0$, then $P(z)$ contains the factor $z - z_1$.

Example $P(z) = z^3 - z^2 + z - 1$. We see that $P(1) = 0$. Indeed, dividing by $z - 1$,

$$\frac{z^3 - z^2 + z - 1}{z - 1} = z^2 + 1.$$

Finally, $$z^3 - z^2 + z - 1 = (z - 1)\overbrace{(z + i)(z - i)}^{z^2+1}$$

LEMMA Suppose that the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers. If z is a root of the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

so is the conjugate number $\bar{z}$.

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†) $a^n - b^n = (a - b)(a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \cdots + a b^{n-2} + b^{n-1})$

Multiply to see this!

Proof: $0 = \overline{0} = \overline{a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0}$

$= \overline{a_n z^n} + \cdots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} = \overline{a}_n \overline{z}^n + \cdots + \overline{a}_2 \overline{z}^2 +$

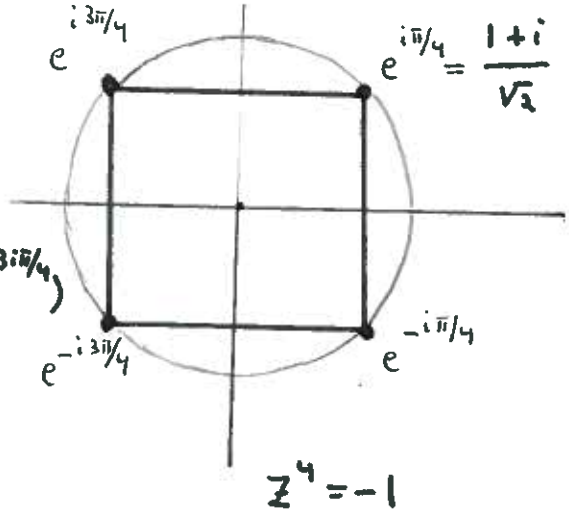
$+ \overline{a}_1 \overline{z} + \overline{a}_0 = a_n \overline{z}^n + \cdots + a_2 \overline{z}^2 + a_1 \overline{z} + a_0 = P(\overline{z})$,

since $\overline{a_k} = a_k$ by assumption. $\square$

$a = \alpha + i\beta, \quad \beta \neq 0$

$$(z-a)(z-\overline{a}) = \underbrace{z^2 - 2\alpha z + \overbrace{\alpha^2+\beta^2}^{=|a|^2}}_{\text{a quadratic factor without real roots}}$$

The conjugate roots combine into a quadratic factor with no real zeros.



Example $z^4 + 1 = 0$

$$z^4+1 = \underbrace{(z-e^{i\pi/4})(z-e^{-i\pi/4})}_{z^2 - z\underbrace{(e^{i\pi/4}+e^{-i\pi/4})}_{\sqrt{2}} + 1}(z-e^{3i\pi/4})(z-e^{-3i\pi/4})$$

$$= (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$$

The factorization $x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$ can be used for the evaluation of the integral $\displaystyle\int \frac{dx}{x^4+1}$.

Read me as "– –"

$$z^4+1 = \left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z + \frac{1+i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right)$$

The equation $z^4 - 1 = 0$ is simpler; $\pm 1, \pm i$.

EXISTENCE OF ROOTS

The FUNDAMENTAL THEOREM OF ALGEBRA was proved by Gauß in 1799. It states that <u>an</u> <u>equation</u> <u>of</u> <u>the</u> n^{th} <u>order</u> always <u>has</u> <u>exactly</u> n <u>complex</u> <u>roots</u>. As a consequence

ROOTS

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = (z-z_1)(z-z_2)\cdots(z-z_n)$$

The same root may be repeated, for example

$$z^3 - 3z^2 + 3z - 1 = (z-1)(z-1)(z-1),$$

that is, $z=1$ is a triple root of $z^3-3z^2+3z-1=0$.

SOME GEOMETRY

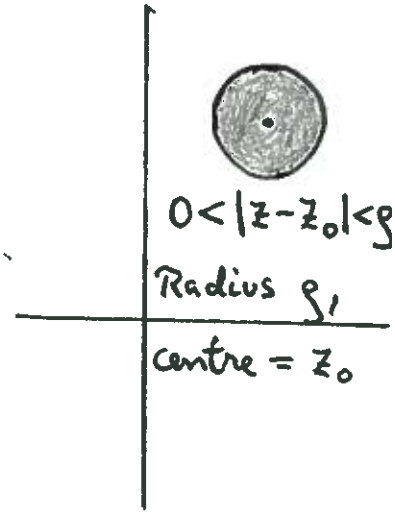
Disc

$|z|<2$

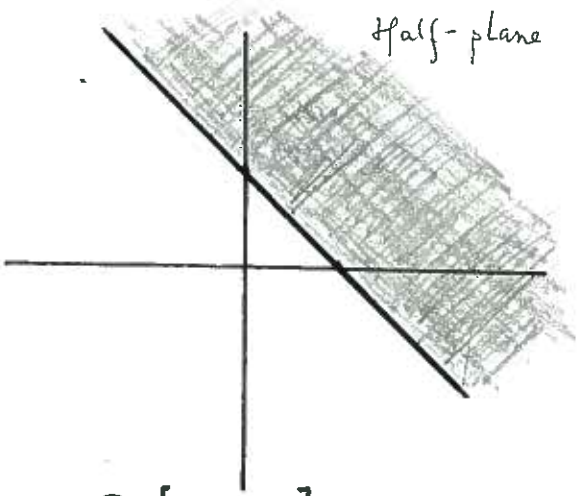
Annulus

$1<|z|<2$

Punctured disc

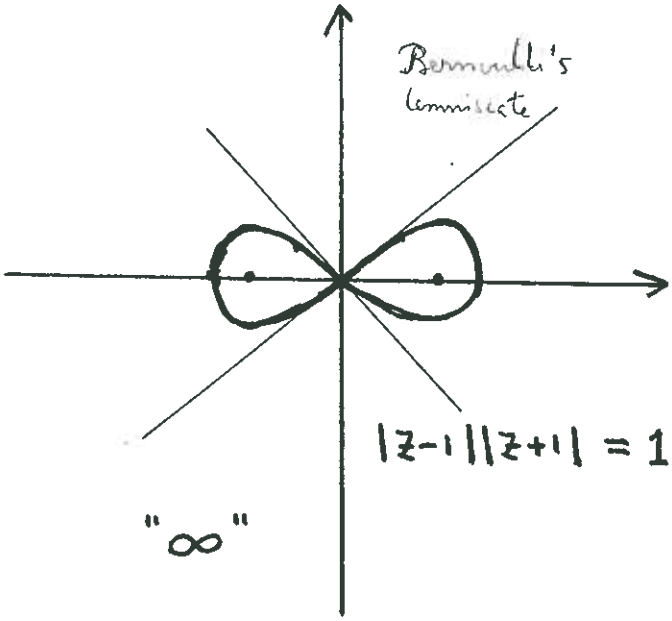


Half-plane



$\mathrm{Re}[z(1-i)] > 1$

Bernoulli's lemniscate



$|z-1||z+1| = 1$

"∞"

ANALYTIC FUNCTIONS

$$\boxed{f(z) = u(x,y) + i\,v(x,y)}$$

<u>Example</u> $f(z) = z^2$. In this example

$$f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{=u(x,y)} + i\,\underbrace{2xy}_{=v(x,y)}.$$

We can also write $f(re^{i\theta}) = r^2 e^{2i\theta}$, $z = re^{i\theta}$.

LIMITS

DEF.: <u>Suppose that</u> $f(z)$ <u>is defined in the punctured disc</u> $0 < |z-a| < \rho$. <u>We say that</u> $f(z) \longrightarrow A$, <u>as</u> $z \longrightarrow a$, <u>if</u>

$$\lim_{z \to a} |f(z) - A| = 0.$$

In other words, given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that

$$|f(z) - A| < \varepsilon, \quad \text{when} \quad 0 < |z-a| < \delta_\varepsilon.$$

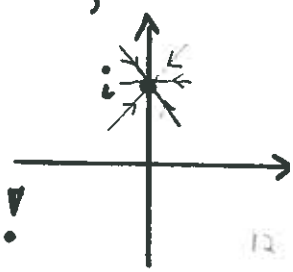
NOTATION $\lim_{z \to a} f(z) = A \iff \lim_{z \to a} |f(z) - A| = 0$

We also write $\lim_{z \to a} f(z) = \infty$, when $\lim_{z \to a} |f(z)| = \infty$.

<u>Example</u> $\lim_{z \to i} \dfrac{z^2+1}{z-i} = 2i$. This is evident, since

$\dfrac{z^2+1}{z-i} = z + i$, when $z \neq i$. NOTICE THAT

Z MAY APPROACH i FROM ANY DIRECTION!



<u>Example</u> $f(z) = \dfrac{z}{|z|}, \quad z \neq 0$

$$\lim_{x \to 0+} f(x) = 1, \quad \lim_{y \to 0+} f(iy) = i, \quad \lim_{x \to 0-} f(x) = -1$$

The values $1, i, -1$ depend on the direction from which the origin is approached. Hence $f(z)$ does not have a limit, as $z \to 0$.

Usual rules for limits of expressions like

$$f(z) \pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)}, \quad F(f(z))$$

are valid.

CONTINUITY

DEF.: <u>Suppose that</u> $f(z)$ <u>is defined in the disc</u> $|z-a| < \rho$. <u>We say that</u> $f(z)$ <u>is continuous at the point</u> a, <u>if</u>

$$\lim_{z \to a} f(z) = f(a).$$

Roughly, this means that $f(z) \approx f(a)$, when $z \approx a$. A small error in a (more exactly, a <u>sufficiently</u> small error in a) leads to only a small error for $f(a)$.

<u>Example</u>

$$g(z) = \begin{cases} \dfrac{z^2+1}{z-i}, & \text{when } z \neq i, \\ 2i, & \text{when } z = i. \end{cases}$$

The function $g(z)$ is continuous at every point.

LEMMA $\lim f(z) = A \iff \lim \mathrm{Re}\, f(z) = \mathrm{Re}\, A$ &
$\lim \mathrm{Im}\, f(z) = \mathrm{Im}\, A$

<u>Proof</u>: To see this, write

$$f = u + iv, \quad A = a + ib.$$

$u = \mathrm{Re}\, f(z)$.

$$|f(z) - A| = \sqrt{(u(x,y) - a)^2 + (v(x,y) - b)^2}. \quad \square$$

THE DERIVATIVE

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

NOTATION

$$\frac{df}{dz} = f'$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

It is important that $\Delta z = \Delta x + i \Delta y$ may approach the origin from any direction.

Given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that

$$\left| f'(z) - \frac{f(z + \Delta z) - f(z)}{\Delta z} \right| < \varepsilon, \quad \text{when} \quad 0 < |\Delta z| < \delta_\varepsilon.$$

Here $|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

<u>Ex</u>: $f(z) = z^2$. $\quad \dfrac{f(z + \Delta z) - f(z)}{\Delta z} =$

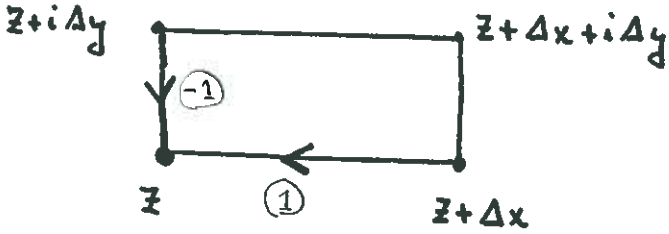
$$= \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2z + \Delta z \longrightarrow 2z; \qquad \frac{dz^2}{dz} = 2z.$$

Ex: $f(z) = \bar{z} = x - iy$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

$$= \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = 1 \neq -1 = \lim_{i\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y}$$



Hence $f(z) = \bar{z}$ is <u>not</u> differentiable at any point.

As a rule of thumb a differentiable function is a function only of the variable z but not of $\bar{z}$ or $|z|$ (notice $|z|^2 = \bar{z}z$).

DIFFERENTIATION RULES:

$$(f+g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dz} f(g(z)) = \left(\frac{df(\zeta)}{d\zeta}\right)_{\zeta = g(z)} \frac{dg(z)}{dz}$$

<u>A differentiable function is continuous.</u>

ANALYTIC FUNCTION

DEFINITION We say that the function $f(z)$ is analytic in the domain Ω, if the derivative $f'(z)$ exists, when $z \in \Omega$.

analytic = holomorphic; conformal mapping = analytic function whose derivative is different from 0

Since z is analytic, so is $z^2 = z\,z$ and so is $z^3 = z^2 \cdot z$ (recall that $(fg)' = f'g + fg'$) and... In general, z^n is analytic in the whole complex plane and

$$\boxed{\frac{dz^n}{dz} = n z^{n-1}} \qquad (n = 1, 2, 3, \ldots)$$

A polynomial $a_0 + a_1 z + \cdots + a_n z^n$ is analytic in the whole complex plane. A rational function

$$\frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m}$$

is analytic, when the points where the denominator is zero (the so called poles) are excluded.

As we will see later, the functions e^z, $\sin z$, $\cos z$, $\tan z$, $\cot z$, $\sinh z$, $\cosh z$, $\tanh z$, $\coth z$, $\ln z$, $\arcsin z$, ... are analytic. One of the miracles about analytic functions is that not only does $f'(z)$ exist but also all higher derivatives $f''(z)$, $f'''(z)$, $f^{(4)}(z)$, ... exist!

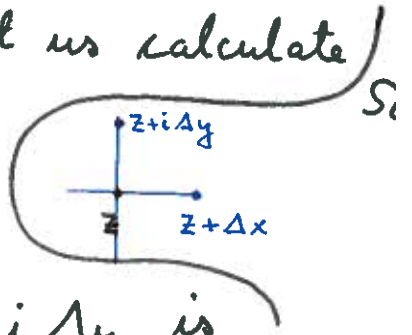
CAUCHY - RIEMANN EQUATIONS

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

THEOREM <u>If</u> $f = u + iv$ <u>is analytic in the domain</u> Ω, <u>then</u> u <u>and</u> v <u>satisfy the Cauchy - Riemann equations in</u> Ω.

<u>Proof</u>: Fix $z = x + iy$ in Ω. Let us calculate

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



in two ways. Recall that $\Delta z = \Delta x + i\Delta y$ is allowed to approach 0 in any way. First, let $\Delta z = \Delta x$. Then

$$\frac{f(z+\Delta x) - f(z)}{\Delta x} = \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\xrightarrow[\Delta x \to 0]{} \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}.$$

Thus

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}}$$

This formula is useful.

Second, let $\Delta z = i\Delta y$. Then

$$\frac{f(z+i\Delta y) - f(z)}{i\Delta y} = \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \cancel{i}\,\frac{v(x, y+\Delta y) - v(x,y)}{\cancel{i}\,\Delta y}$$

$$\xrightarrow[i\Delta y \to 0]{} \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Thus

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Equating the two expressions for $f'(z)$ we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

This yields the Cauchy-Riemann equations. $\square$

<u>Example</u> $f(z) = \frac{1}{z} = \frac{x-iy}{x^2+y^2}$, $z = x+iy \neq 0$.

The functions

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

satisfy the Cauchy-Riemann equations. Indeed

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{\partial v}{\partial y}, \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{2xy}{(x^2+y^2)^2} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

We can also verify that

$$\underline{f'(z)} = -\frac{1}{z^2} = -\frac{\bar{z}^2}{|z|^4} = -\frac{(x-iy)^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+2ixy}{(x^2+y^2)^2} = \underline{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}.$$

<u>Example</u> Suppose that $f(z)$ is analytic and that $|f(z)|$ is constant. Then $f(z)$ is constant.

$$f = u + iv, \quad \boxed{u^2 + v^2 \equiv C^2 \text{ (a constant)}}$$

Assumption.

$$\begin{cases} 2uu_x + 2vv_x = 0 \\ 2uu_y + 2vv_y = 0 \end{cases}$$

After differentiation. Here $u_x = \frac{\partial u}{\partial x}, \ldots$

$$\begin{cases} (1) \quad uu_x - vu_y = 0 \\ (2) \quad uu_y + vu_x = 0 \end{cases}$$

By the Cauchy-Riemann equations

$$u^2 u_x \overset{(1)}{=} uvu_y \overset{(2)}{=} -v^2 u_x, \quad (u^2+v^2)u_x = 0$$

We get

$$\underline{(u^2+v^2)u_x = 0}, \quad \underline{(u^2+v^2)u_y = 0}.$$

If $C = 0$, $u = v = 0$ and $f(z) \equiv 0$. If $C \neq 0$, then $u_x \equiv 0$ and $u_y \equiv 0$ and so $u \equiv$ constant. By the Cauchy-Riemann equations $v_x \equiv 0$ and $v_y \equiv 0$ and so $v \equiv$ constant. This proves that $f(z)$ is a constant.

If the Cauchy-Riemann equations hold, then we have an analytic function. More precisely:

THEOREM <u>Suppose that the first partial derivatives</u> u_x, u_y, v_x, v_y <u>are continuous and satisfy the Cauchy-Riemann equations</u> $u_x = v_y$, $u_y = -v_x$ <u>in the domain</u> Ω. <u>Then the function</u> $f = u + iv$ <u>is analytic in</u> Ω.

<u>Proof</u>: Skipped!

SOME ANALYTIC FUNCTIONS

$$\boxed{e^z = e^x(\cos y + i \sin y),} \qquad z = x + iy$$

$$\frac{de^z}{dz} = e^z, \quad |e^z| = e^x, \quad e^z \neq 0$$

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

$$e^z = \lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n \qquad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{z + 2\pi i} = e^z \quad (\text{period } 2\pi i)$$

$e^{\pi\sqrt{163}} = 262537412640768743,999999999999...$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\boxed{\sin^2 z + \cos^2 z = 1}$$

THE DIFFERENTIATION RULES ARE THE FAMILIAR FORMULAS $(\sin z)' = \cos z, \ldots$

$$\begin{cases} \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1 \\ \cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \end{cases}$$

$$\sin(z + 2\pi) = \sin z \quad (\text{period } 2\pi)$$

! $|\sin x| \leq 1$ but it is possible that $|\sin z| > 1$. !

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$\frac{355}{113} = 2.$ $\quad 3^3 + 4^4 + 3^3 + 5^5 = 2.$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{2z} - 1}{e^{2z} + 1}, \quad \coth z$$

$$\boxed{\cosh^2 z - \sinh^2 z = 1} \qquad \frac{d\cosh z}{dz} = \sinh z$$

$$\cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z,$$
$$\cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z$$

<u>Example</u> Starting with the familiar formula

$$\sin^2 z + \cos^2 z = 1$$

we get, replacing z by iz,

$$\underbrace{\sin^2(iz)}_{(i \sinh z)^2} + \underbrace{\cos^2(iz)}_{(\cosh z)^2} = 1,$$

that is $\cosh^2 z - \sinh^2 z = 1$.

$$e^{w} = z, \ z \neq 0 \overset{\text{DEF.}}{\Longleftrightarrow} w = \overset{\log z}{\ln z}$$

The notation $\log z$ is frequently used

<u>Preliminary remark</u>: if $e^{w} = z$, then $e^{w + 2n\pi i} = z$. Thus $\ln z$ is defined only up to a multiple of $2\pi i$.

$$w = u + iv, \quad z = re^{i\theta}$$

$$e^{w} = z \Longleftrightarrow e^{u} e^{iv} = re^{i\theta} \Longleftrightarrow r = e^{u} \ \& \ v = \theta + 2n\pi$$

$$w = \ln r + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\boxed{\ln z = \ln|z| + i\arg(z)}$$

<u>Examples</u> $\ln(-1) = i(\pi + 2n\pi)$

$\ln(i) = i\left(\frac{\pi}{2} + 2n\pi\right)$

$\ln 1 = 0 + 2n\pi i$ Usually, we take $n = 0$.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \quad (\text{modulo } 2\pi i)$$

DEF.: $a^z = e^{z \ln a}$ $(a \neq 0)$.

<u>Examples</u> $i^i = e^{i \ln i} = e^{-\frac{\pi}{2}} e^{\pm 2n\pi}$

$\sqrt[n]{z} = z^{\frac{1}{n}} = e^{\frac{1}{n}\ln z}$

REMARK The only analytic function which coincides with e^x on the real axis is e^z. No other function will do. More generally, we have the PRINCIPLE OF ANALYTIC CONTINUATION: Suppose that the functions $f(z)$ and $g(z)$ are analytic and that $f(x) = g(x)$ on the real axis. Then $f(z) = g(z)$ (in the domain of analyticity).

¿ $1 = (-1)^2 = \sqrt{(-1)^2} = \sqrt{-1\cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$?

¿ $2\log(-1) = \log(-1) + \log(-1) = \log[-1\cdot -1] = \log 1 = 0 \Rightarrow$
$\log(-1) = 0 \Rightarrow e^0 = -1$. But $e^0 = +1$, as we know! ?

The complex **exponential function**

$$e^z = \exp z = e^x(\cos y + i \sin y) \tag{5}$$

is periodic with $2\pi i$, reduces to e^x if $z = x$ ($y = 0$), and has the derivative e^z.

The **trigonometric functions** are

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \end{aligned} \tag{6}$$

$\tan z = (\sin z)/\cos z$, $\cot z = 1/\tan z$, etc.

The **hyperbolic functions** are

$$\begin{aligned} \cosh z &= \frac{1}{2}(e^z + e^{-z}) = \cos iz, \\ \sinh z &= \frac{1}{2}(e^z - e^{-z}) = -i \sin iz, \end{aligned} \tag{7}$$

etc. An **entire function** is a function that is analytic everywhere in the complex plane. The functions in (5)–(7) are entire.

The **natural logarithm** is

$$\begin{aligned} \ln z &= \ln |z| + i \arg z && (\arg z = \theta,\ z \neq 0) \\ &= \ln |z| + i \operatorname{Arg} z \pm 2n\pi i && (n = 0, 1, \cdots), \end{aligned} \tag{8}$$

where $\operatorname{Arg} z$ is the **principal value** of $\arg z$, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** $\operatorname{Ln} z$ of $\ln z$; thus

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z. \tag{8*}$$

General powers are defined by

$$z^c = e^{c \ln z} \qquad (c \text{ complex},\ z \neq 0). \tag{9}$$

LAPLACE'S EQUATION $\Delta u = 0$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

<u>The real and imaginary parts of an analytic function satisfy the Laplace equation:</u>

$$f = u + iv; \quad \Delta u = 0, \quad \Delta v = 0$$

This follows from the Cauchy-Riemann equations, if one already knows that u and v has continuous second partial derivatives.

The solutions of Laplace's equation are called <u>harmonic functions</u>. They are important in <u>potential theory</u> (gravitation, electricity, ...).

<u>Ex.</u>: $\Delta\left(\frac{x}{x^2+y^2}\right) = 0$, when $(x,y) \neq (0,0)$. $\mathrm{Re}\,\frac{1}{z}$!

If $\Delta u = 0$, we can construct (at least locally) a function v such that $u + iv$ is analytic. (We say that u and v are <u>conjugate</u> harmonic functions.)

EXAMPLE $u(x,y) = 2x(1-y)$. $v(x,y) = ?$

$\Delta u = u_{xx} + u_{yy} = 0$ Hence u is harmonic.

Important!

Cauchy-Riemann $\begin{cases} v_y = u_x = 2(1-y) \iff v = 2y - y^2 + \varphi(x) \\ v_x = -u_y = \underline{2x} \qquad\qquad v_x = 0 + \underline{\varphi'(x)} \end{cases}$

Hence $2x = \varphi'(x)$, $\varphi(x) = x^2 + C$. Thus $\underbrace{v = 2y + x^2 - y^2 + C}_{\text{ANSWER}}$

As a matter of fact

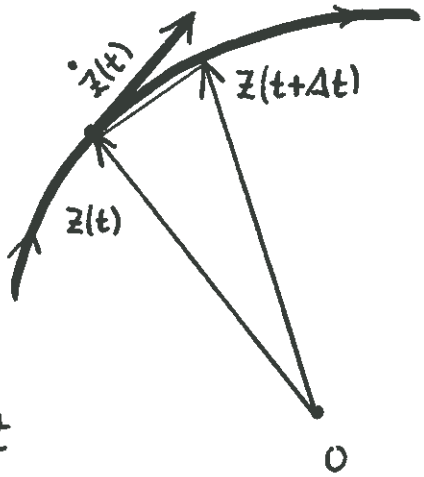
$$f(z) = u + iv = iz^2 + 2z + iC$$

INTEGRALS

<u>Curves</u> $\quad z(t) = x(t) + i\,y(t) \qquad t = \text{"time"}$

$$\dot{z}(t) = \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

$$= \cdots = \dot{x}(t) + i\,\dot{y}(t)$$



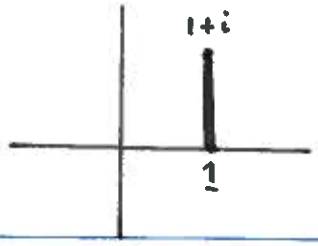
$$|\dot{z}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \quad \text{"SPEED"}$$

$$\text{length of the curve} = \int_{t=\alpha}^{\beta} |\dot{z}(t)|\,dt$$

<u>Examples</u>

1°) $\quad z = 1 + ti, \quad 0 \le t \le 1$

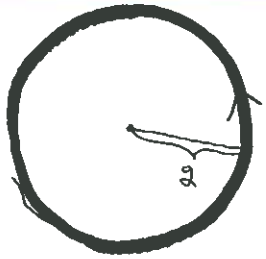
$\quad \dot{z} = i \qquad$ A <u>segment</u>.



2°) $\quad z(t) = 2\cos t + 2i\sin t = 2e^{it}$

$\quad \dot{z}(t) = -2\sin t + 2i\cos t = 2ie^{it}$

$\quad x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4$

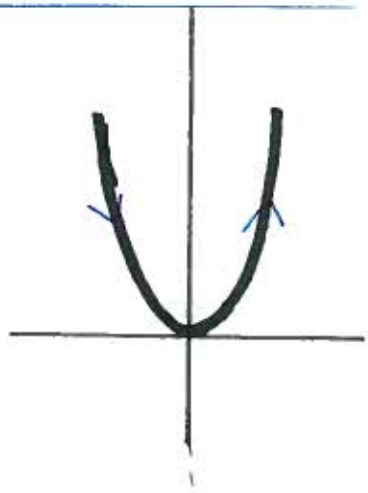


A <u>circle</u> of radius 2.

3°) $\quad z_1(t) = t + it^2, \quad -\infty < t < \infty$

$\quad z_2(t) = \ln t + i(\ln t)^2, \quad 0 < t < \infty$

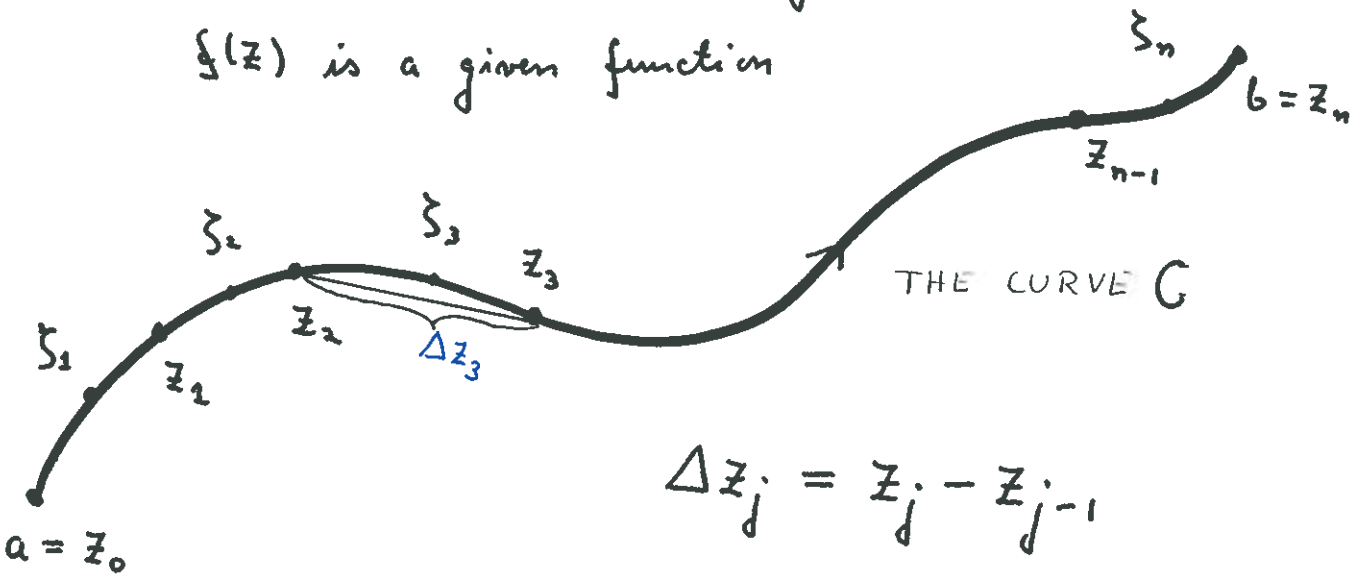
Both represent the same curve, the <u>parabola</u> $y = x^2$.



Curve integrals (line integrals)

$\int_C f(z)\,dz$

$f(z)$ is a given function



$$\Delta z_j = z_j - z_{j-1}$$

1°) Pick arbitrary points $a = z_0, z_1, z_2, z_3, \ldots, z_{n-1}, z_n = b$ in successive order, dividing the curve into n arcs, the so called subarcs.

2°) Select a point ζ_j from the arc between z_{j-1} and z_j, $j = 1, 2, 3, \ldots, n$

3°) Form the sum (Riemann's sum)

$$f(\zeta_1)(z_1 - z_0) + f(\zeta_2)(z_2 - z_1) + \cdots + f(\zeta_n)(z_n - z_{n-1})$$

$$= \sum_{j=1}^{n} f(\zeta_j)\Delta z_j$$

Suppose that the division is done so that the length of the longest subarc (and hence the lengths of all subarcs) approaches zero. Under suitable assumptions the Riemann sums will approach a limit

that is independent of the division:

$$\lim \sum f(\zeta_j)\Delta z_j \overset{\text{NOTATION}}{=} \int_C f(z)\,dz$$

a continuous function defined at all points of the curve.

a piecewise smooth curve

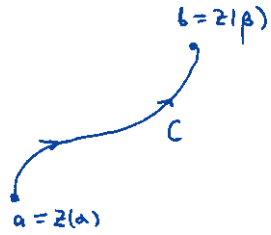
This is <u>the</u> <u>integral</u> <u>of</u> $f(z)$ <u>along</u> <u>the</u> <u>curve</u> C.

If the curve C has the parametrization

$$z = z(t) = x(t) + i\,y(t), \quad \alpha \le t \le \beta,$$

and if $\dot{z}(t)$ is continuous, we can use the formula

$$\boxed{\int_C f(z)\,dz = \int_\alpha^\beta f(z(t))\,\dot{z}(t)\,dt}$$



for the calculation. (As a rule of thumb, the integrand $f(z(t))\dot{z}(t)$ should be continuous.)

<u>Ex.</u>

$$\boxed{\int_{|z|=1} \frac{dz}{z} = 2\pi i}$$

The unit circle, counterclockwise.

$$z = e^{it}, \quad dz = i e^{it} \qquad \frac{dz}{z} = \frac{i e^{it}\,dt}{e^{it}} = i\,dt$$

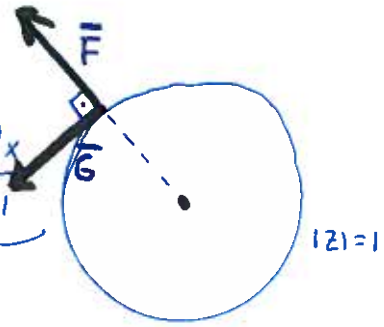
$$\oint \frac{dz}{z} = \int_0^{2\pi} i\,dt = 2\pi i.$$ A similar calculation yields

$$\oint z^n\,dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n = 0, 1, \pm 2, \pm 3, \ldots, \end{cases}$$

again counterclockwise along the unit circle $|z| = 1$.

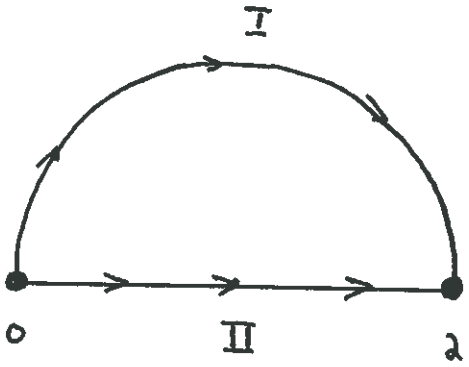
Interpretation of $\oint \frac{dz}{z} = 2\pi i$.

$$2\pi i = \oint \frac{dx + i\,dy}{x + iy} = \underbrace{\oint_{x^2+y^2=1} \frac{x\,dx + y\,dy}{x^2+y^2}}_{=0} + i \underbrace{\oint_{x^2+y^2=1} \frac{x\,dy - y\,dx}{x^2+y^2}}_{=2\pi}$$



The work done by the force $\bar{F} = x\hat{\imath} + y\hat{\jmath}$ is 0 and the work done by the force $\bar{G} = -y\hat{\imath} + x\hat{\jmath}$ is 2π. (Notice that $x^2 + y^2 = 1$.)

EXAMPLE The integral $\int \bar{z}\,dz$ from 0 to 2 along the half-circle is $2 - i\pi$, and along the line segment it is 2. Indeed,



I) $z(t) = 1 + e^{it}$, half-circle

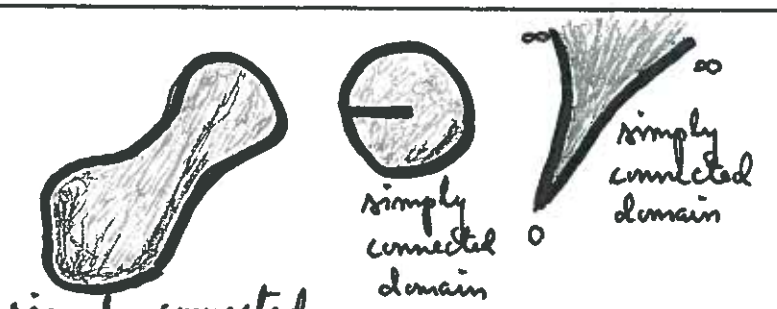
$dz = i e^{it} dt$.

$$\int_{\frown} \bar{z}\,dz = \int_{\pi}^{0} (1 + e^{-it})\, i e^{it} dt = \int_{\pi}^{0} i\left[e^{it} + 1\right] dt$$

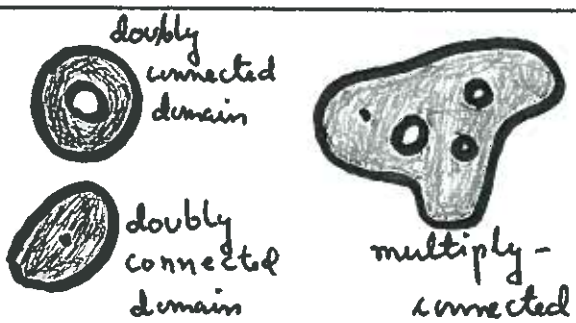
$$= \Big/_{\pi}^{0} e^{it} + it = 2 - i\pi.$$

II) $z = x$, $dz = dx$; $\bar{z} = x$ line segment

$$\int_{\text{—}} \bar{z}\,dz = \int_0^2 x\,dx = \Big/_0^2 \frac{x^2}{2} = 2.$$



SIMPLY CONNECTED



NOT!

The ML-inequality

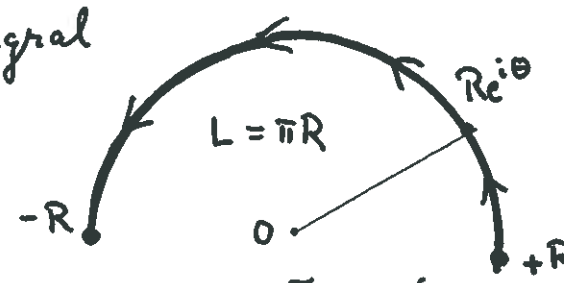
$$\boxed{\begin{aligned} &\left| \int_C f(z)\,dz \right| \le ML \\ &L = \int_C |dz| = \text{length of the curve } C \\ &|f(z)| \le M, \text{ when } z \in C \end{aligned}}$$

is useful. (The function $f(z)$ does not have to be analytic here.)

Examen 8. des. 1997 Ex. 56.

<u>Example</u> Show that the integral

$$\int \frac{e^{3iz}\,dz}{z^2+2z+10}$$



taken along the arc $z = Re^{i\theta}$, $0 \le \theta \le \pi$, of a half-circle approaches zero as $R \longrightarrow +\infty$.

$$e^{3iz} = e^{3i(R\cos\theta + iR\sin\theta)} = e^{-3R\sin\theta}\, e^{3iR\cos\theta}$$

$$|e^{3iz}| = e^{-3R\sin\theta} \le 1, \text{ when } 0 \le \theta \le \pi.$$

$$|z^2+2z+10| \ge |z^2| - |2z+10| \ge R^2 - 2R - 10$$

$$\left| \frac{e^{3iz}}{z^2+2z+10} \right| \le \frac{1}{R^2-2R-10}. \text{ By ML-ineq.}$$

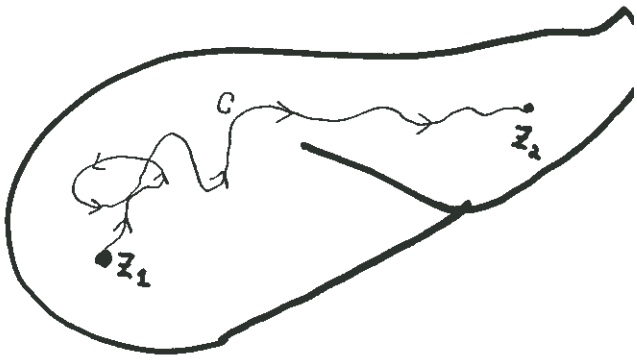
$$\left| \int \cdots dz \right| \le \frac{\pi R}{R^2-2R-10} \longrightarrow 0, \text{ as } R \longrightarrow +\infty.$$

Integrals of analytic functions in SIMPLY connected domains can be evaluated in the following way.

THEOREM <u>Let</u> $f(z)$ <u>be analytic in a simply connected domain</u> Ω. <u>Then there exists a function</u> $F(z)$ <u>such that</u> $F'(z) = f(z)$ <u>when</u> $z \in \Omega$. <u>Moreover,</u>

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

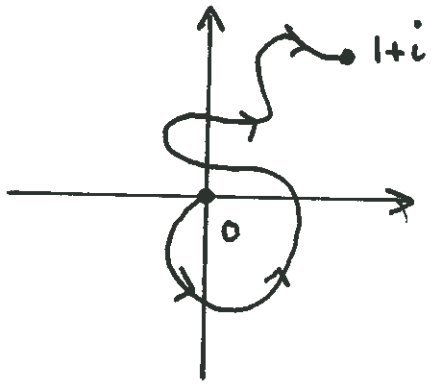
<u>where the integral is taken along any path in</u> Ω <u>joining</u> z_1 <u>and</u> z_2.



The integral depends only on the endpoints z_1 and z_2 but is independent of the path. The proof of the theorem is based on Cauchy's Integral Theorem, a fundamental result, which will be derived later.

EXAMPLES

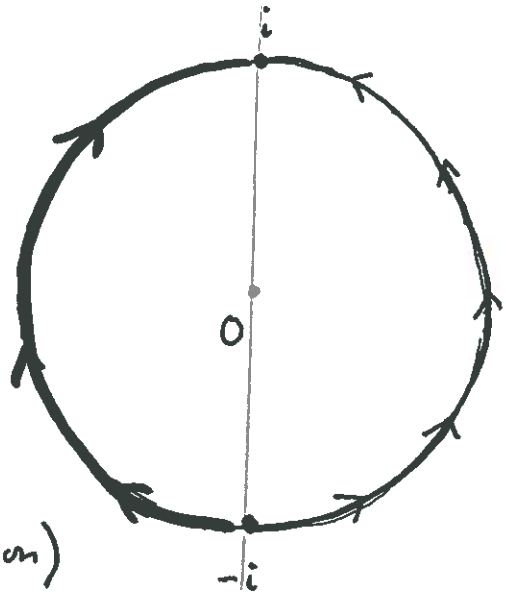
$$\int_0^{1+i} z^2\,dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{(1+i)^3}{3} = \frac{2i-2}{3}$$



$$\int_{z_1}^{z_2} \sin z\,dz = -\cos z_2 + \cos z_1$$

EXAMPLE The integral

$$\int_{-i}^{i} \frac{dz}{z}$$ depends on the



path between the endpoints $\pm i$.

Direct calculation (write $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ and so on) yields

$$\int_{-i}^{i} \frac{dz}{z} = \begin{cases} \pi i & \text{along the right half-circle} \\ -\pi i & \text{along the left half-circle} \end{cases}$$

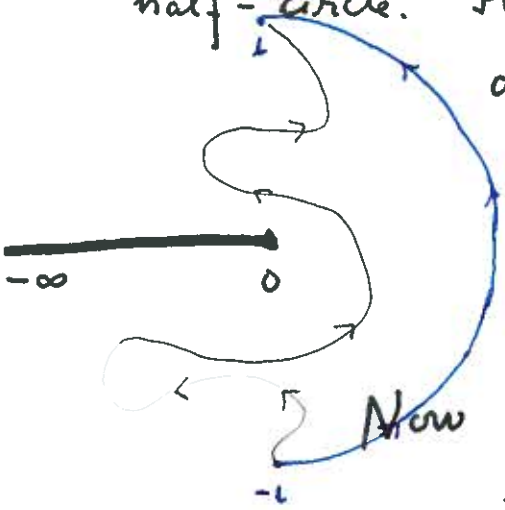
Hence the formula

$$\int_{-i}^{i} \frac{dz}{z} = \ln i - \ln(-i)$$

Recall that $\frac{d \ln z}{dz} = \frac{1}{z}$

cannot be used without some precaution

In general, we have to specify a <u>simply</u> connected domain not containing the origin ($1/z$ is analytic except at the origin) but containing the desired half-circle. For instance, removing the negative real axis (a slit from 0 to $-\infty$), a simply connected domain remains.



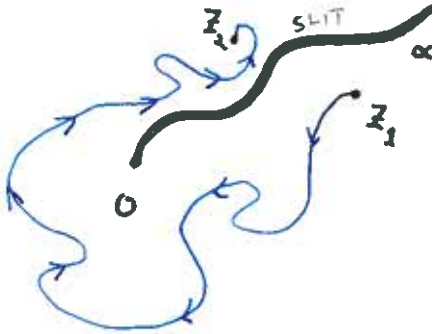
$$\ln z = \ln r + i\theta + 2ni\pi, \quad n = 0$$

Now $$\int_{-i}^{i} \frac{dz}{z} = \ln i - \ln(-i) = i\frac{\pi}{2} - \left(-i\frac{\pi}{2}\right) = i\pi$$

$-\pi < \theta < +\pi$ along any path not crossing the slit $(-\infty, 0]$. In particular, for

the right (not the left) half-circle.

REMARK. Usually, the functions $\ln z$ and $\sqrt{z}$ require a slit from zero to infinity. This prevents winding around 0.



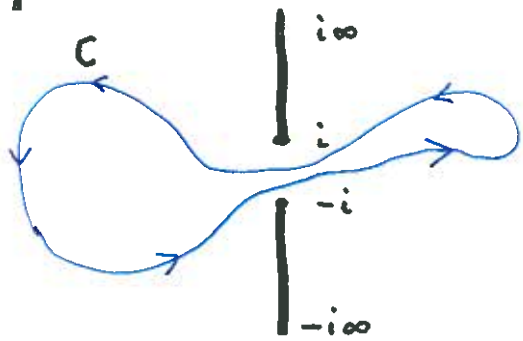
CAUCHY'S INTEGRAL THEOREM

THEOREM *If $f(z)$ is analytic in a simply connected domain Ω, then for every closed path (= "contour") in Ω*

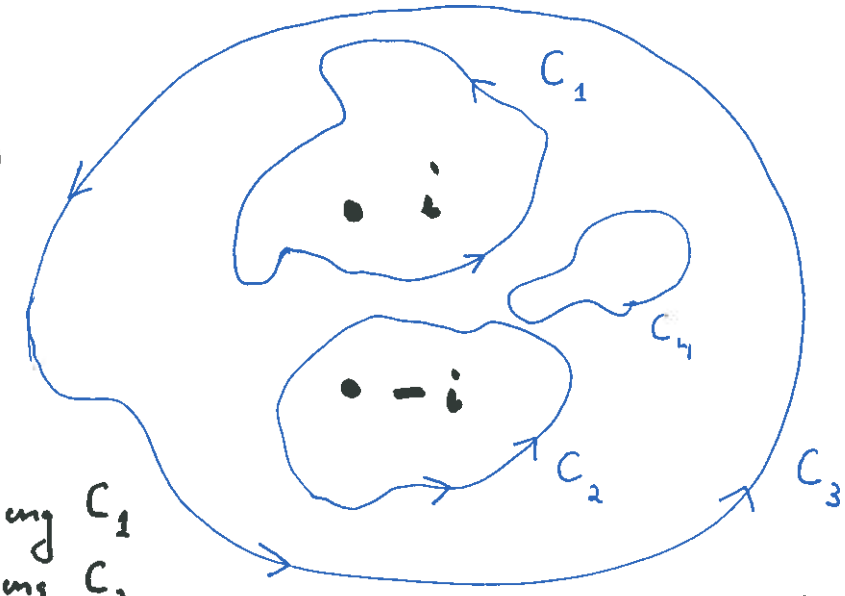
$$\oint f(z)\,dz = 0 .$$

Ex. $\oint e^z dz = 0$, $\oint \sin z\, dz = 0$, $\oint z^3 dz = 0$, $\oint e^{\sin z} dz = 0$. The contour is arbitrary.

Ex $\displaystyle\oint_C \frac{dz}{1+z^2} = 0$ provided that the curve does not enclose $\pm i$.



A precise calculation (later!) reveals that the integral

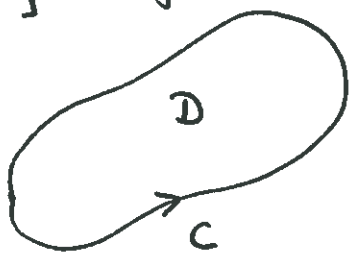


$$\oint \frac{dz}{1+z^2} = \begin{cases} \pi & \text{along } C_1 \\ -\pi & \text{along } C_2 \\ 0 & \text{along } C_3 \\ 0 & \text{along } C_4 \end{cases}$$

Cauchy's Integral Thm can be directly applied only for the path C_4.

<u>Proof</u> of Cauchy's Integral Theorem under the additional assumption that $f'(z)$ is continuous: Recall Green's formula

$$\int_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy$$



Here D is the domain bounded by C. We have

$$\oint_C f(z)\,dz = \oint (u+iv)(dx+i\,dy) = \oint_C u\,dx - v\,dy$$

$$+ i \oint_C u\,dy + v\,dx \underset{\text{formula}}{\overset{\text{GREEN's}}{=}} \iint_D \left[\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$$

$$+ i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy \underset{\text{RIEMANN}}{\overset{\text{CAUCHY}}{=}} \iint_D 0\,dx\,dy \; +$$

$$+ i \iint_D 0 \, dx\,dy = 0 + i\,0,$$ where the Cauchy-Riemann equations were used:

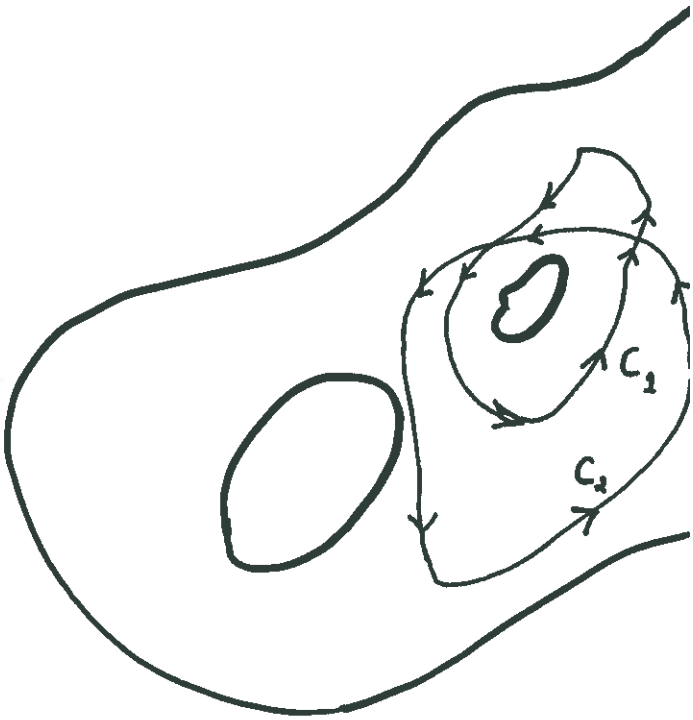
$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0. \quad \square$$

REMARK. About 1900 Goursat removed the assumption about the continuity of the derivative $f'(z)$. He merely assumed that $f'(z)$ exists, i.e., that $f(z)$ is analytic, in his proof. One often calls the result the CAUCHY-GOURSAT theorem.

PRINCIPLE OF DEFORMATION OF PATH. In multiply connected domains this is of actual interest.

If the closed curve C_1 can be "continuously deformed" into C_2 without ever touching the boundary of the domain of analyticity for $f(z)$, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

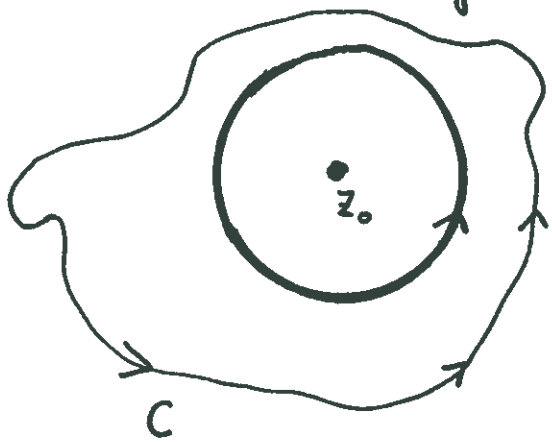


For example

$$\oint_C \frac{dz}{z-z_0} = \begin{cases} \boxed{2\pi i}, & \text{if } C \text{ encloses } z_0 \\ 0, & \text{if } z_0 \text{ is outside the domain bounded by } C. \end{cases}$$

The integrand $1/(z-z_0)$ is not analytic, when $z = z_0$.

If C encloses z_0, then we can deform C to a circle centered at z_0.



Thus

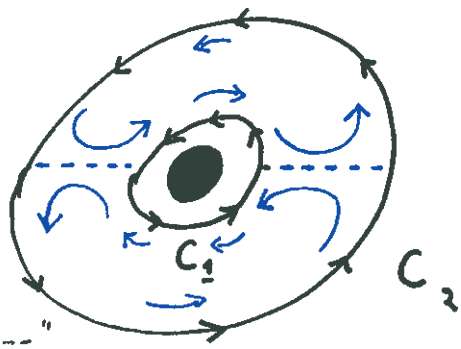
$$\oint_C \frac{dz}{z-z_0} = \oint_{\substack{\text{Circle} \\ |z-z_0|=1}} \frac{dz}{z-z_0} = \boxed{2\pi i}$$

For the circle the evaluation of the integral is easy.

The other case follows from Cauchy's theorem:

Now $\dfrac{1}{z-z_0}$ is analytic in the domain bounded by C, indeed.

The idea is:



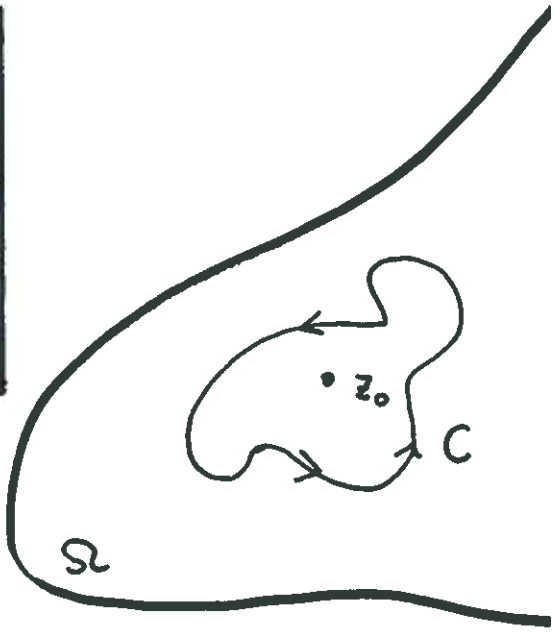
Because of cancellations and reversions of direction

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Here $f(z)$ is analytic in the domain between the curves C_1 and C_2. The segments "- - - -" divide this domain into two simply connected domains, where Cauchy's Int. Theorem can be applied.

CAUCHY'S INTEGRAL FORMULA:

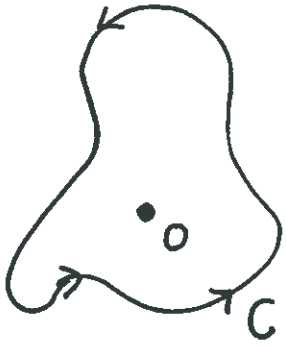
$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}}$$



LET $f(z)$ BE ANALYTIC IN A SIMPLY CONNECTED DOMAIN Ω. SUPPOSE THAT z_0 IS A POINT IN Ω. IF THE CLOSED CURVE C LIES IN Ω AND ENCLOSES z_0, THEN THE FORMULA ABOVE HOLDS.

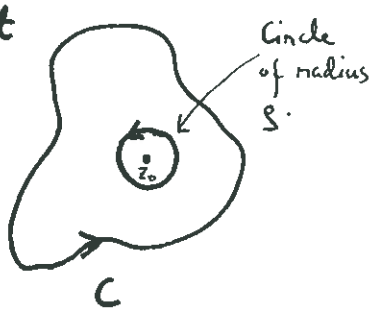
EXAMPLE

$$1 = e^0 = \frac{1}{2\pi i} \oint_C \frac{e^z\,dz}{z - 0}$$



<u>Proof</u>: The integrand $\dfrac{f(z)}{z - z_0}$ is analytic except at the point z_0. Deforming the path C to a small circle $|z - z_0| = \rho$, we get

$$\oint_C \frac{f(z)\,dz}{z - z_0} = \int_{|z - z_0| = \rho} \frac{f(z)\,dz}{z - z_0}$$



Writing

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

we arrive at

$$\oint_C \frac{f(z)\,dz}{z - z_0} = \underbrace{f(z_0) \oint_{\text{Circle}} \frac{dz}{z - z_0}}_{= 2\pi i\, f(z_0)} + \oint_{\text{Circle}} \frac{f(z) - f(z_0)}{z - z_0}\, dz$$

As a matter of fact, the second integral is zero. Indeed, given an arbitrarily small number $\varepsilon > 0$, $|f(z) - f(z_0)| < \varepsilon$ when $|z - z_0| = \rho$ is small enough. (This is the meaning of $f(z_0) = \lim_{z \to z_0} f(z)$.). Then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \frac{\varepsilon}{\rho}$$

$|z - z_0| = \rho$

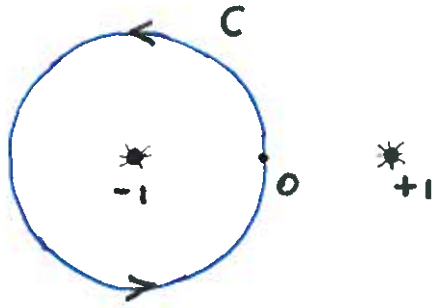
and

$$\left| \oint \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \frac{\varepsilon}{\rho} \cdot 2\pi\rho = 2\pi\varepsilon .$$

But $\varepsilon > 0$ is as small as we please. Hence the second integral vanishes. – What is left is Cauchy's Formula. $\square$

Ex. Integrate $\dfrac{e^z}{z^2-1}$

along the circle $|z+1| = 1$.



poles: ± 1

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

$$\oint_C \frac{e^z\,dz}{z^2-1} = \underbrace{\frac{1}{2}\oint_C \frac{e^z\,dz}{z-1}}_{\substack{=0 \text{ because} \\ \frac{e^z}{z-1} \text{ is analytic when} \\ |z+1| \le 1.}} - \frac{1}{2}\oint_C \frac{e^z\,dz}{\underbrace{z+1}_{z-(-1)}} = 0 - \frac{2\pi i}{2}e^{-1}$$

$$= -\frac{\pi i}{e}.$$

DERIVATIVES An analytic function has derivatives of all orders. Recall that in the definition merely the first derivative is required. This can be read off from Cauchy's formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2}$$

$$f''(z) = \frac{2!}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^3}$$

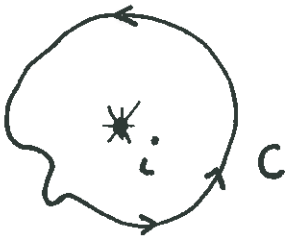
$$\vdots$$

In general,

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}}$$

Ex. $$\oint_C \frac{\sin z\, dz}{(z-i)^3} = \frac{2\pi i}{2!}\left[\frac{d^2 \sin z}{dz^2}\right]_{z=i}$$

$$= -\pi i \sin(i) = \pi \sinh(1)$$



LIOUVILLE'S THEOREM If $f(z)$ is analytic and bounded in the whole complex plane, then $f(z) \equiv$ Constant.

<u>Proof</u>: By assumption $|f(z)| \le M$, when all z. Fix a point z_0 and integrate along the circle $|z - z_0| = R$ in the formula

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz .$$

$$\left| \frac{1}{2\pi i} \frac{f(z)}{(z-z_0)^2} \right| = \frac{|f(z)|}{2\pi R^2} \leq \frac{M}{2\pi R^2}$$

on the circle. By the "ML-inequality

$$|f'(z_0)| \leq \frac{M}{2\pi R^2} \cdot 2\pi R = \frac{M}{R}.$$

As $R \longrightarrow +\infty$ this estimate yields that $f'(z_0) = 0$. Since z_0 was an arbitrary point, this shows that $f'(z) \equiv 0$. Hence $f(z) \equiv$ Constant. Indeed,

$$f(z) = f(0) + \int_0^z \overbrace{f'(z)}^{\equiv 0} dz = f(0) + 0. \qquad \square$$

Liouville's Theorem can be used to prove the Fundamental Theorem of Algebra: <u>Every polynomial</u>

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

<u>has at least one zero</u>. The reasoning goes as follows. If $P(z) \neq 0$ for all z (antithesis), then

$$f(z) = \frac{1}{P(z)}$$

is analytic in the whole complex plane (we never were to divide by 0). One can show that $f(z)$ is bounded in the whole complex plane. By Liouville's Theorem $f(z)$ is a constant. This is absurd. Thus there is a zero.

POWER SERIES

TAYLOR
(MACLAURIN)
LAURENT

For a series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

we form the partial sums

$$S_1 = z_1, \quad S_2 = z_1 + z_2, \quad S_3 = z_1 + z_2 + z_3, \ldots,$$

$$S_n = \underbrace{z_1 + z_2 + \cdots + z_n}_{n \text{ terms}}, \ldots.$$

If there is a complex number S such that $\lim_{n\to\infty} S_n = S$, then we say that the series $\sum z_n$ is <u>convergent</u> and that S is its <u>sum</u>. This is denoted by

$$S = \sum_{n=1}^{\infty} z_n .$$

In all other cases the series is called <u>divergent</u>. For example, the harmonic series $\sum \frac{1}{n}$ is divergent. Indeed,

Not a "number".

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = +\infty$$

LEMMA $\sum |z_n| < \infty \Rightarrow \sum z_n$ is convergent.

<u>absolutely</u> convergent!

THE GEOMETRIC SERIES $1 + z + z^2 + \cdots$

$$\underbrace{1 + z + z^2 + \cdots + z^{n-1}}_{n \text{ terms are added up}} = \frac{1 - z^n}{1 - z}, \qquad z \neq 1$$

$$1 + z + z^2 + \cdots = \frac{1}{1 - z}, \quad \text{when } |z| < 1$$

The geometric series diverges, when $|z| \geq 1$. We say that $R = 1$ is the <u>radius of convergence</u>. The following manipulations are valid, when $|z| < 1$. By differentiation

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5 + \cdots$$

$$\frac{1!}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

$$\frac{2!}{(1-z)^3} = 2 + 3\cdot 2z + 4\cdot 3z^2 + 5\cdot 4z^3 + \cdots$$

$$\frac{3!}{(1-z)^4} = 3\cdot 2 + 4\cdot 3\cdot 2z + 5\cdot 4\cdot 3z^2 + \cdots$$

$$\vdots$$

and by integration

$$\ln \frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

Writing $z = re^{i\theta}$, $0 \le r < 1$,

$$1 + re^{i\theta} + r^2 e^{2i\theta} + r^3 e^{3i\theta} + \cdots = \frac{1}{1 - re^{i\theta}}$$

and taking real and imaginary parts we obtain the interesting sums

$$1 + r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}$$

$$r\sin\theta + r^2\sin 2\theta + r^3\sin 3\theta + \cdots = \frac{r\sin\theta}{1 + r^2 - 2r\cos\theta}$$

both valid, when $0 \le r < 1$.

POWER SERIES

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0) = a_0 + a_1(z - z_0) +$$

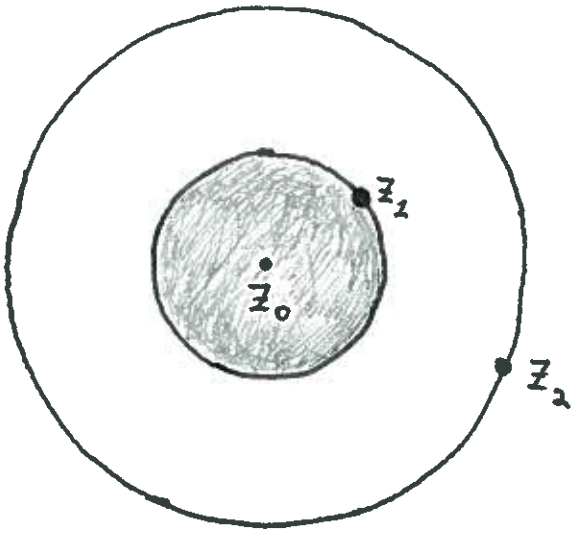
$$+ a_2(z - z_0)^2 + \cdots + \underbrace{\overset{\text{Coefficient}}{a_{n-1}}(z - z_0)^{n-1}}_{\text{The } n^{\text{th}} \text{ term}} + \cdots\cdots$$

Often the "center" $z_0 = 0$.

ABEL'S THEOREM. <u>If the power series converges at the point $z = z_1 \neq z_0$, then it converges (even absolutely) at all points z for which $|z - z_0| < |z_1 - z_0|$.</u>

<u>If the power series diverges at the point</u>

z_2, <u>then it diverges at all points</u> z <u>for which</u> $|z - z_0| > |z_1 - z_0|$.



Convergence inside of the small circle.

Divergence outside of the large circle.

Abel's theorem implies the existence of a so called RADIUS OF CONVERGENCE.

! To each power series $\sum a_n (z - z_0)^n$ there corresponds a radius of convergence R, $0 \leq R \leq +\infty$ such that the series converges (even absolutely), when $|z - z_0| < R$ and diverges when $|z - z_0| > R$. (If $R = 0$, the series diverges, when $z \neq z_0$. If $R = \infty$, the series converges for all z.)

Please, notice that nothing is said about points on the circle of convergence $|z - z_0| = R$.

$\sum n!\, z^n$	$\sum z^n$	$\sum \frac{z^n}{n!}$
$R = 0$	$R = 1$	$R = +\infty$
Converges only for $z = 0$	Geometric series	Always convergent. e^z

$$\boxed{R = \frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}, \qquad R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}}$$

EXAMPLES $\sum \frac{z^n}{n!}$. $\quad \frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} =$

$$= \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 \; ; \quad R = \text{"}\frac{1}{0}\text{"} = +\infty .$$

$$\underbrace{\sum n^{-1998} z^n, \; \sum n^{-1} z^n, \; \sum z^n, \; \sum n^{1998} z^n}_{\substack{R = 1 \text{ for all} \\ \text{four !}}}$$

Strictly inside the circle of convergence we can differentiate and integrate a power series termwise.

EXAMPLE $S(z) = \sum\limits_{n=1}^{\infty} n z^n = ?$. $R = 1$

$$S(z) = 1\cdot z + 2z^2 + 3z^3 + \cdots \qquad |z| < 1$$

$$\frac{S(z)}{z} = 1 + 2z + 3z^2 + \cdots$$

$$= \frac{d}{dz}\left\{ z + z^2 + z^3 + \cdots \right\} = \frac{d}{dz}\left(\frac{z}{1-z}\right)$$

$$= \frac{1}{(1-z)^2} \; ; \qquad S(z) = \frac{z}{(1-z)^2}$$

$S(0) = 0$ as it should be.

$$1z + 2z^2 + 3z^3 + \cdots = \frac{z}{(1-z)^2}, \text{ when } |z| < 1$$

The function

$$f(z) = \sum a_n (z - z_0)^n$$

is analytic, when $|z - z_0| < R$ = radius of convergence. Moreover

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

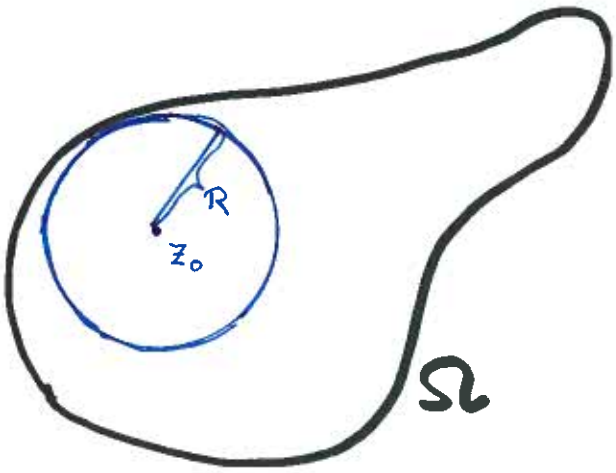
Actually, all analytic functions are power series! More precisely:

THEOREM <u>Let</u> $f(z)$ <u>be analytic in the domain</u> Ω <u>and let</u> z_0 <u>be any interior point in</u> Ω. <u>Then the expansion</u>

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

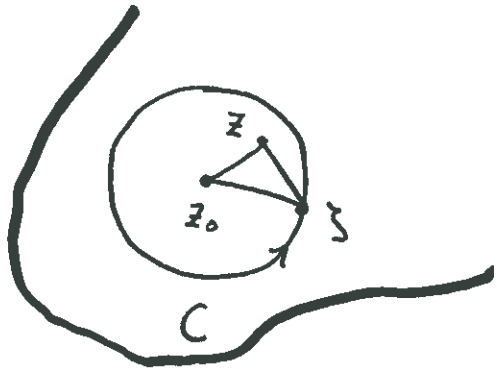
<u>is valid in the largest disc contained in</u> Ω <u>with center</u> z_0.

$|z - z_0| < R$ = the distance from z_0 to the boundary of the domain Ω.



<u>Proof</u>: Start with Cauchy's Formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$



$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \qquad \left|\frac{z - z_0}{\zeta - z_0}\right| < 1$$

$$= \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]} \overset{\text{GEOM. SERIES}}{=} \frac{1}{\zeta - z_0}\left\{1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots\right\}$$

$$f(z) = \overbrace{\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\,d\zeta}^{= f(z_0)} + \frac{z - z_0}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta +$$

$$+ \frac{(z - z_0)^2}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta + \cdots$$

$$= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2$$

$$+ \cdots$$

Recall the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta. \quad \square$$

The expansion is called the TAYLOR (MACLAURIN) series.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\tan z = z + \frac{z^3}{3} + \frac{2 \cdot z^5}{15} + \frac{17 \cdot z^7}{315} + \frac{62 \cdot z^9}{2835} + \cdots$$

when $|z| < \pi/2$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1$$

$(\log 1 = 0)$

$$\overline{\arctan} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots, \quad |z| < 1$$

$$\log \frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\right), \quad |z| < 1$$

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \frac{p(p-1)(p-2)}{3!} z^3 + \cdots$$

Converges at least, when $|z| < 1$.

MANIPULATIONS:

$$\sum a_n z^n \pm \sum b_n z^n = \sum (a_n \pm b_n) z^n$$

when $|z| < \min\{R_a, R_b\}$ = the smaller radius of convergence

Cauchy's rule:

$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = a_0 b_0 + (a_1 b_0 + a_0 b_1) z +$$

$$+ (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots + (a_n b_0 + a_{n-1} b_1 +$$

$$+ a_{n-2} b_2 + \cdots + a_1 b_{n-1} + a_0 b_n) z^n + \cdots ,$$

$|z| < \min(R_a, R_b)$

$$b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$$

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

$$\begin{array}{llll}
a_0 b_0 + & a_0 b_1 z + & a_0 b_2 z^2 + & a_0 b_3 z^3 + \cdots \\
 & a_1 b_0 z + & a_1 b_1 z^2 + & a_1 b_2 z^3 + \cdots \\
 & & a_2 b_0 z^2 + & a_2 b_1 z^3 + \cdots \\
 & & & a_3 b_0 z^3 + \cdots
\end{array}$$

ADD UP

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) z +$$

Ex. $\frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z} = (1 + z + z^2 + z^3 + \cdots) \cdot$

$(1 + z + z^2 + z^3 + \cdots) = 1 \cdot 1 + (1 \cdot 1 + 1 \cdot 1) z + (1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) z^2$

$+ \cdots = 1 + 2z + 3z^2 + 4z^3 + \cdots, \quad |z| < 1.$

<u>Ex</u>: $\tan z = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < \frac{\pi}{2}$

Use the formula

$$\frac{d \tan z}{dz} = 1 + \tan^2 z$$

to calculate the coefficients:

$$a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots =$$

$$1 + a_1^2 z^2 + 2a_1 a_2 z^3 + (2a_1 a_3 + a_2^2) z^4 + \cdots$$

Comparing coefficients:

$a_1 = 1, \quad 2a_2 = 0, \quad 3a_3 = a_1^2, \quad 4a_4 = 2a_1 a_2,$

$5a_5 = 2a_1 a_3 + a_2^2.$

$a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{1}{3}, \quad a_4 = 0,$

$a_5 = 2/15$

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots$$

Accidentally, we have solved the differential equation

$$y' = 1 + y^2, \quad y(0) = 0.$$

THE LAURENT SERIES

The expansion

$$\frac{\sin z}{z^5} = z^{-4} - \frac{1}{6} z^{-2} + \frac{1}{120} - \frac{1}{5040} z^2 + \cdots,$$

$0 < |z| < \infty$, is not a Maclaurin (= Taylor, $z_0 = 0$) series, since it contains negative powers. It is a LAURENT series.

THEOREM <u>Suppose that $f(z)$ is analytic in the annulus $R_1 < |z - z_0| < R_2$. Then $f(z)$ can be represented by the Laurent series</u>

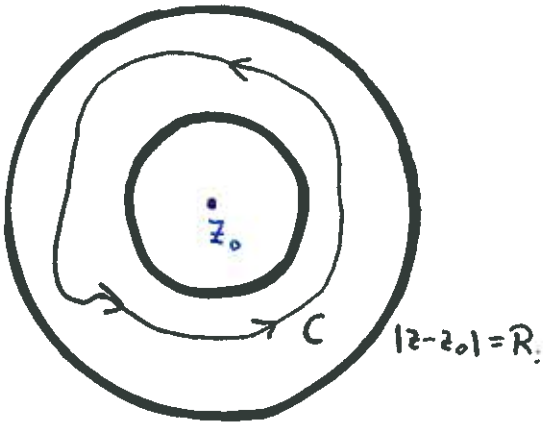
$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 +$$

$$+ a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

<u>when</u> $R_1 < |z - z_0| < R_2$

<u>Moreover</u>,

$$\boxed{a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz}$$



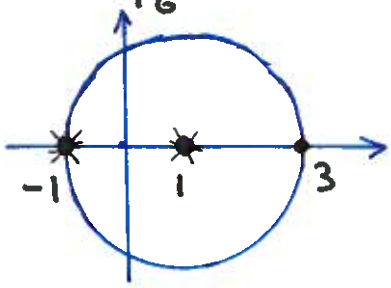
<u>where C is a closed curve in the annulus.</u>

REMARK: The case $R_1 = 0$ is the most significant. Then $f(z)$ is analytic in the punctured disc $0 < |z - z_0| < R_2$.

!

Ex. $\dfrac{1}{z^2-1} = \dfrac{1/2}{z-1} - \dfrac{1}{4} + \dfrac{1}{8}(z-1) - \dfrac{1}{16}(z-1)^2$

$+ \cdots$, when $0 < |z-1| < 2$.



$$\frac{1}{z^2-1} = \frac{1}{z-1}\left[\frac{1}{2+(z-1)}\right] = \frac{1}{2(z-1)\left[1-\left(-\frac{z-1}{2}\right)\right]}$$

$$= \frac{1}{2(z-1)}\left\{1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\right\}$$

<u>Ex.</u>: $e^{\frac{1}{z}} = 1 + \dfrac{1}{z} + \dfrac{1}{2!\,z^2} + \dfrac{1}{3!\,z^3} + \cdots$, $z \neq 0$.

We say that $f(z)$ has a POLE of the nth order at z_0, if the Laurent expansion has the form

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0)$$

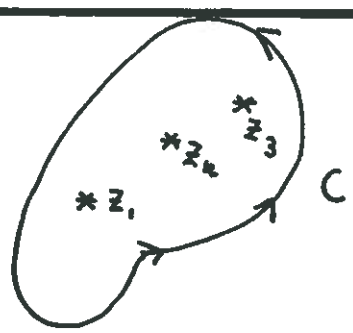
$$+ a_2(z-z_0)^2 + \cdots,$$

$a_{-n} \neq 0$, when $0 < |z-z_0| <$ some radius. If the Laurent expansion has infinitely many terms with negative powers, we talk about an essential singularity. For example, the origin is an essential singularity for $e^{1/z}$.

CALCULUS OF RESIDUES

BASIC RESULT

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$



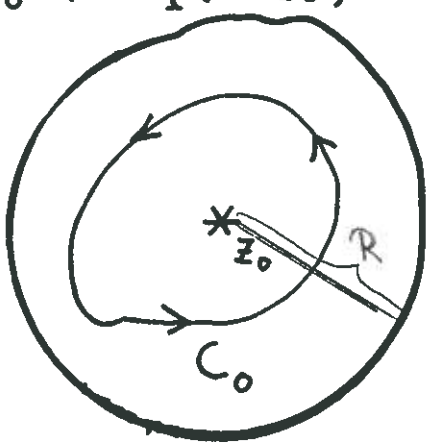
Here $f(z)$ is analytic on and inside C, except at the points $z_1, z_2, \ldots, z_n$

Let us begin with a simpler situation. Suppose that we have the Laurent expansion

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0)$$

$$+ a_2(z-z_0)^2 + \cdots$$

when $0 < |z-z_0| < R$. (Near z_0)



Integrating termwise we get

$$\oint_{C_0} f(z)\,dz = 2\pi a_{-1},$$

since

$$\oint_C \frac{dz}{(z-z_0)^k} = 0, \quad k = 0, -1, \pm 2, \ldots$$

and

$$\oint_C \frac{dz}{z-z_0} = 2\pi i.$$

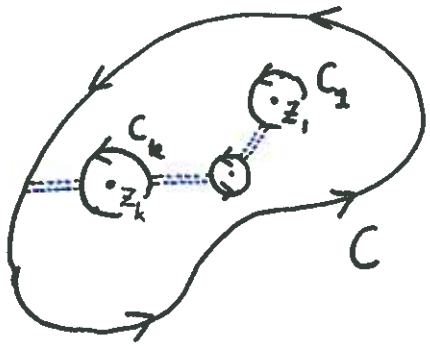
We say that the coefficient a_{-1} is the RESIDUE of $f(z)$ at the point z_0:

$$\boxed{a_{-1} = \operatorname*{Res}_{z=z_0} f(z)}$$

Returning to the general case with several singularities, we have

$$\oint_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$

where C_k is a small circle around the singular point z_k. The circles are not allowed to overlap.



$$\int_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

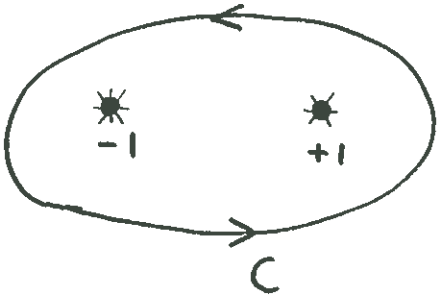
$$= 2\pi i \operatorname*{Res}_{z=z_1} f(z) + \cdots + 2\pi i \operatorname*{Res}_{z=z_n} f(z)$$

This is the desired formula for evaluating the integral in terms of residues.

$169 = 13^2$, $961 = 31^2$.

EXAMPLE

$$\oint_C \frac{dz}{1-z^2} = 2\pi i \left\{ \operatorname*{Res}_{z=1} \frac{1}{1-z^2} + \operatorname*{Res}_{z=-1} \frac{1}{1-z^2} \right\}$$



$\underline{z=1}$ $\quad \dfrac{1}{1-z^2} = \dfrac{1}{(1-z)(1+z)}$

$$= \frac{1}{(1-z)(2+(z-1))} = \frac{-1/2}{(z-1)\left(1+\frac{z-1}{2}\right)} =$$

$0 < |z-1| < 2$

$$= -\frac{1}{2(z-1)} \left\{ 1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \cdots \right\}$$

$$= \frac{-1/2}{z-1} + \frac{1}{4} - \frac{1}{8}(z-1) + \cdots \; ; \quad \operatorname*{Res}_{z=1} \frac{1}{1-z^2} = \underline{-\frac{1}{2}}$$

$\underline{z=-1}$ In a similar manner, when $0 < |z+1| < 2$,

$$\frac{1}{1-z^2} = \frac{1/2}{z+1} - \frac{1}{4} + \frac{1}{8}(z+1) + \cdots \; ; \quad \operatorname*{Res}_{z=-1} \frac{1}{1-z^2} = \underline{\frac{1}{2}}$$

<u>Answer</u>: $\displaystyle\oint_C \frac{dz}{1-z^2} = 2\pi i \left\{ \frac{1}{2} - \frac{1}{2} \right\} = 0.$

At a SIMPLE pole z_0

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \left[(z-z_0) f(z) \right]}$$

At a pole of the second order

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right].$$

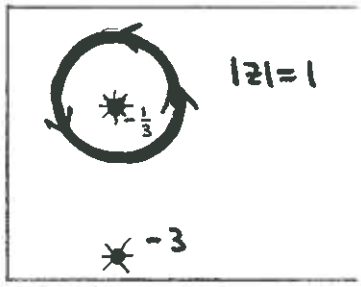
INTEGRALS $\displaystyle\int_0^{2\pi} \underbrace{R(\sin\theta, \cos\theta)}_{\text{A rational expression in } \sin\theta \text{ and } \cos\theta.} d\theta = I$

$$z = e^{i\theta}, \quad dz = i e^{i\theta} d\theta = i z \, d\theta, \quad d\theta = \frac{dz}{iz}$$

$$\left.\begin{aligned} \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right) \\ \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right) \end{aligned}\right\}$$

This transforms the integral to a curve integral along the <u>unit</u> <u>circle</u> $|z| = 1$.

Ex. $$I = \int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \oint_{|z|=1} \frac{\frac{dz}{iz}}{5 + \frac{3}{2}\left(z + \frac{1}{z}\right)}$$



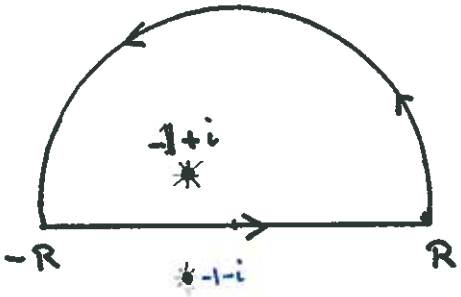
$$= \cdots = -\frac{2i}{3} \oint_{|z|=1} \frac{dz}{\left(z + \frac{1}{3}\right)(z+3)}$$

$$= -\frac{2i}{3} \cdot 2\pi i \operatorname*{Res}_{z=-\frac{1}{3}} \frac{1}{\left(z+\frac{1}{3}\right)(z+3)} = \frac{4\pi}{3} \cdot \frac{1}{-\frac{1}{3} + 3} = \frac{\pi}{2}$$

Notice that the pole at $z = -3$ is not included, since it is outside the contour $|z| = 1$.

INTEGRALS LIKE $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

Try to integrate over a large half-circle and let the radius go to $+\infty$.



$\oint f(z)\,dz$

Ex. $\displaystyle I = \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

$$\int_{-R}^{R} \frac{dx}{x^2+2x+2} + \int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{R^2e^{2i\theta}+2Re^{i\theta}+2} = \oint \frac{dz}{z^2+2z+2}$$

CALCULATE!

$$= 2\pi i \operatorname*{Res}_{z=-1+i}\left(\frac{1}{z^2+2z+2}\right) = \frac{2\pi i}{2i} = \pi$$

Poles: $-1 \pm i$. Only $-1+i$ is in the upper half-plane.

On the half-circle $z = Re^{i\theta}$

$$|z^2+2z+2| \geq |z^2| - |2z+2| \geq R^2 - 2R - 2$$

By the ML-inequality

$$\left|\int_{\cap} \cdots dz\right| \leq \frac{\pi R}{R^2-2R-2} \xrightarrow[R\to+\infty]{} 0$$

Result: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$.

<u>Remark</u> If $P(z)$ and $Q(z)$ are polynomials such that 1°) $Q(x) \neq 0$ (no zeros on the real axis) and 2°) degree $Q \geq 2 +$ degree P, then the integral over the half-circle vanishes as its radius goes to ∞. We are left with the formula

$$\int_{-\infty}^{\infty} \underbrace{\frac{P(x)}{Q(x)}}_{\text{Rational function where the degree of the denominator is at least two units higher than the degree of the numerator.}} dx = 2\pi i \underbrace{\sum_{y>0} \operatorname{Res} \frac{P(z)}{Q(z)}}_{\text{The sum of the residues in the upper half-plane}}$$

<u>Ex.:</u> $$\int_0^{\infty} \frac{1+x^2}{1+x^4} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4} dx$$

$$\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4} dx = 2\pi i \left\{ \operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{1+z^2}{1+z^4} + \operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{1+z^2}{1+z^4} \right\}$$

There are four simple poles, viz. $\frac{\pm 1 \pm i}{\sqrt{2}}$ (the roots of $1+z^4 = 0$), but only two of them are in the upper half-plane. Hence only the residues at $\pm \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ are included in the calculation. Please, calculate the residues needed yourself!

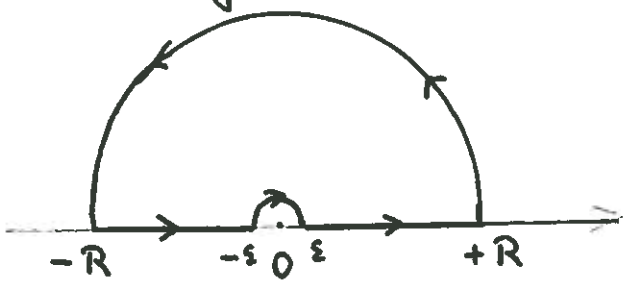
$1233 = 12^2 + 33^2$, $8833 = 88^2 + 33^2$

EX. $$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

The function $f(z) = \dfrac{\sin z}{z}$ will <u>not</u> work here! Consider instead

$$\frac{e^{iz}}{z} = \frac{\cos z + i \sin z}{z} \qquad (z \neq 0)$$

and integrate along the contour



The half-circles have radii ε and R, $0 < \varepsilon < R$

$$0 = \oint \frac{e^{iz}dz}{z} = \int_{-R}^{-\varepsilon} \frac{e^{ix}dx}{x} + \int_{\varepsilon}^{R} \frac{e^{ix}dx}{x} + \int_{\substack{|z|=R \\ y>0}} \frac{e^{iz}dz}{z}$$

$$+ \int_{|z|=\varepsilon} \frac{e^{iz}}{z}\,dz$$

On the large half-circle

$$z = Re^{i\theta}, \quad \left|\frac{e^{iz}}{z}\right| = \frac{e^{-R\sin\theta}}{R}, \quad \frac{dz}{z} = i\,d\theta$$

$$\left|\int_{\substack{|z|=R \\ y>0}} \frac{e^{iz}dz}{z}\right| \leq \int_0^\pi e^{-R\sin\theta}\,d\theta \xrightarrow[\text{as } R\to\infty]{} 0.$$

Thus

$$0 = \underbrace{\int_{-\infty}^{-\varepsilon} \frac{e^{ix}\,dx}{x} + \int_{\varepsilon}^{\infty} \frac{e^{ix}\,dx}{x}}_{i\int_{-\infty}^{-\varepsilon} \frac{\sin x}{x}\,dx + i\int_{\varepsilon}^{\infty} \frac{\sin x}{x}\,dx} + \underbrace{\int_{\substack{z=\varepsilon e^{i\theta} \\ \pi \geq \theta \geq 0}} \frac{e^{iz}}{z}\,dz}_{\longrightarrow -\pi i}$$

$e^{ix} = \cos x + i \sin x$

By symmetry, $\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{\cos x}{x}\,dx = 0.$

The integral along the small half-circle is

$$\int_{\curvearrowright} \frac{e^{iz}\,dz}{z} = \underbrace{\int \frac{dz}{z}}_{= -\pi i} + \underbrace{\int \frac{iz}{z}\,dz}_{2\varepsilon i} + \underbrace{\int \frac{i^2 z^2}{2!}\,dz}_{-\frac{\varepsilon^3}{3}} + \cdots$$

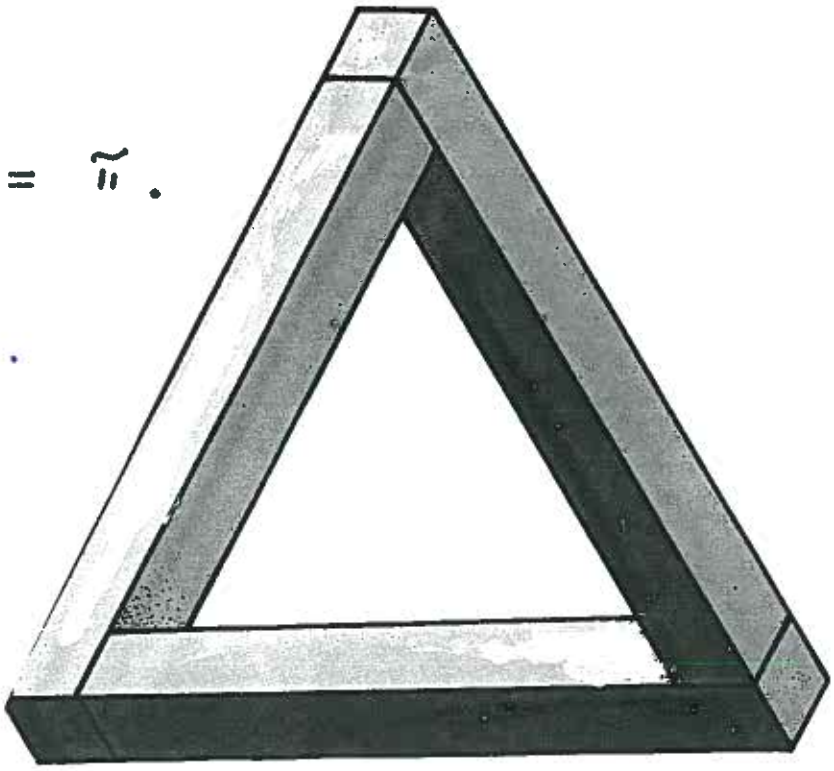
$$\xrightarrow[\varepsilon \to 0+]{} -\pi i \qquad (\text{clockwise !})$$

The result is

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi .$$

Remark:

$$\int_{-\infty}^{\infty} \left|\frac{\sin x}{x}\right| dx = +\infty$$



That is, how it is !

THE FOURIER TRANSFORM

$$\begin{cases} \hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \\[2ex] f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{+i\omega x}\, d\omega \end{cases}$$

<u>Notation</u>: $\mathcal{F}(f(x)) = \hat{f}(\omega)$. <u>Ex.</u> $\mathcal{F}(e^{-ax^2}) = \dfrac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2a}}$.

$$\mathcal{F}(af(x) + bg(x)) = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)) \qquad \text{LINEARITY}$$

<u>Convolution</u>: $(f * g)(x) = \displaystyle\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$ (DEFINITION)

$$\boxed{\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f)\mathcal{F}(g)}$$

<u>Example</u> $\mathcal{F}\left(\dfrac{1}{1+x^2}\right) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \dfrac{e^{-i\omega x}}{1+x^2}\, dx$

$= \cdots = \sqrt{\frac{\pi}{2}}\, e^{-|\omega|}$. Integrate along a large half-circle (keep $\omega \cdot \operatorname{Im} z \le 0$); residues.

The contour, if $\omega > 0$.