Problem 1 Find all complex numbers z such that $z^2 = 2\bar{z}$ and plot them on the complex plane.

Solution We write z in polar form, $z = re^{i\theta}$. Then $z^2 = r^2 e^{2i\theta}$ and $\bar{z} = re^{-i\theta}$. The equation $z^2 = 2\bar{z}$ gives $r^2 = 2r$ and $2\theta = -\theta + 2\pi k$. First, we get that either $r = 0$ or $r = 2$. Further, $3\theta = 2\pi k$ and $\theta = 2\pi k/3$. We get the following solutions

$$z_1 = 0,\ z_2 = 2,\ z_3 = 2e^{2\pi/3},\ z_4 = 2e^{4\pi/3}.$$

Problem 2 Solve the initial value problem

$$y'' - 2y' + y = 3t + 25\sin(3t), \quad y(0) = 1,\ y'(0) = 3$$

Solution First we solve the corresponding homogeneous equation $y'' - 2y' + y = 0$. The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$. It has a double root, $\lambda_1 = \lambda_2 = 1$. The fundamental system of solutions of the homogeneous equation is $y_1(t) = e^t$ and $y_2(t) = te^t$.

To find a particular solution for non-homogeneous equation we use the method of undetermined coefficients. The right hand side is $3t + 25\sin(3t)$ then we look for a solution of the form

$$y_p(t) = at + b + c\cos(3t) + d\sin(3t).$$

Do determine the coefficients, we compute the derivatives and sett them in the equation, we obtain:

$$-9c\cos(3t) - 9d\sin(3t) - 2a + 6c\sin(3t) - 6d\cos(3t) + at + b + c\cos(3t) + d\sin(3t) = 3t + 25\sin(3t).$$

We compare the coefficients on the both sides of the equation and get

$$-2a + b = 0,\ a = 3, \quad -8c - 6d = 0,\ -8d + 6c = 25.$$

Then $a = 3, b = 6, c = 3/2, d = -2$. We obtain a particular solution

$$y_p(t) = 3t + 6 + 3/2\cos(3t) - 2\sin(3t).$$

The general solution is $y(t) = y_p(t) + y_h(t) = y_p(t) + c_1 e^t + c_2 te^t$.

Finally, we use the initial conditions to specify the coefficients c_1 and c_2. We have

$$y'(t) = 3 - 9/2\sin(3t) - 6\cos(3t) + c_1e^t + c_2(e^t + te^t),$$

$$y(0) = 6 + 3/2 + c_1,\ y'(0) = 3 - 6 + c_1 + c_2$$

Then $c_1 = -6.5,\ c_2 = 12.5$. We have

$$y(t) = 3t + 6 + 3/2\cos(3t) - 2\sin(3t) - 6.5e^t + 12.5te^t.$$

Problem 3

a) Find two linearly independent solutions of the homogeneous differential equation

$$y'' - 3y' + 2y = 0$$

Solution The characteristic polynomial is $\lambda^2 - 3\lambda + 2 = 0$. It has two roots $\lambda_1 = 2$ and $\lambda_2 = 1$. We have two linearly independent solutions $y_1 = e^{2t}$ and $y_2 = e^t$.

b) Find the general solution of the differential equation

$$y'' - 3y' + 2y = \cos(e^{-t})$$

(Hint: $\int e^{-(k+1)t}\cos(e^{-t})dt = -\int u^k \cos u\, du, \quad u = e^{-t}$.)

Solution We use the method of variation of parameters. From (a) we know two solutions $y_1 = e^{2t}$ and $y_2 = e^t$. The Wronski determinant is

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = -e^{3t}.$$

Then

$$y_p = -y_1\int \frac{r(t)y_2(t)}{W(t)}dt + y_2\int \frac{r(t)y_1(t)}{W(t)}dt$$

We have $r(t) = \cos(e^{-t})$. The first integral is

$$-\int \frac{r(t)y_2(t)}{W(t)}dt = \int \cos(e^{-t})e^{-2t}dt = -\int u\cos u du$$

where $u = e^{-t}$. Integrating by parts, we get

$$-\int u\cos u du = -u\sin u - \cos u = -e^{-t}\sin e^{-t} - \cos e^{-t}$$

The second integral is simpler,

$$\int \frac{r(t)y_1(t)}{W(t)}dt = -\int \cos(e^{-t})e^{-t}dt = \int \cos u du = \sin e^{-t}$$

Finally,

$$y_p = -e^{2t}\cos(e^{-t}), \quad \text{and} \quad y(t) = y_p(t) + y_h(t) = C_1e^t + (-\cos(e^{-t}) + C_2)e^{2t}.$$

Problem 4

a) Find all solutions of the system

$$\begin{aligned} x_1 + 4x_2 + x_3 &= 0 \\ 5x_1 + 22x_2 + 8x_3 &= 0 \\ -3x_1 - 6x_2 + 6x_3 &= 0 \end{aligned}$$

Solution We perform Gauss elimination of the coefficient matrix

$$\begin{bmatrix} 1 & 4 & 1 \\ 5 & 22 & 8 \\ -3 & -6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus x_3 is a free variable and the system has infinitely many solutions parametrized by

$$x_3 = t,\ x_2 = -1.5t,\ x_1 = 5t$$

b) Are the vectors

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 22 \\ -6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$$

linearly independent?

Solution The vectors above are the column vector of the matrix A from part (a). We saw that the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. It means that the coefficient matrix A is singular and the column vectors are linearly dependent.

Problem 5 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 + 8v_2 \\ v_1 + 4v_2 \\ 2v_1 - 4v_2 \end{bmatrix}$$

a) Show that T is a linear transformation and find its standard matrix.

Solution To check that T is linear we compute $T(a\mathbf{v} + b\mathbf{w})$ and show that it is equal to $aT(\mathbf{v}) + bT(\mathbf{w})$. We have $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = [3, 1, 2]^T$ and $T(\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) =$ $[8, 4, -4]^T$. Then

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 8 \\ 1 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

b) Is T one-to-one? Is T onto?

Solution T is one-to-one if $\mathbf{v} \neq \mathbf{w}$ implies $T(\mathbf{v}) \neq T(\mathbf{w})$. It is equivalent to $T(\mathbf{v}) = \mathbf{0}$ only for $\mathbf{v} = \mathbf{0}$. We see that if $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \mathbf{0}$ then

$$3v_1 + 8v_2 = 0, v_1 + 4v_2 = 0, 2v_1 - 4v_2 = 0.$$

The last two equations are not proportional and imply $v_1 = v_2 = 0$. Thus T is one-to-one.

T is onto, means that for every $\mathbf{u} \in \mathbb{R}^3$ there is $\mathbf{v} \in \mathbb{R}^2$ such that $T(\mathbf{v}) = \mathbf{u}$. But $T : \mathbb{R}^2 \to \mathbb{R}^3$ and the image of T is a subspace of dimension at most 2, it can not be the whole $\mathbb{R}^3$. Thus T is not onto.

Problem 6 Let

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 6 \\ -3 & -2 & 7 \end{bmatrix}$$

a) Find the LU factorization of M.

Solution First we do Gauss elimination

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 6 \\ -3 & -2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 2 \\ 0 & 7 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 27 \end{bmatrix}$$

Thus U is the final matrix in the Gauss elimination and L can be read from the Gauss elimination by normalizing the pivot columns

$$U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 27 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7 & 1 \end{bmatrix}$$

b) Compute the determinant of M.

Solution We have $M = LU$, then $\det(M) = \det(L)\det(U)$. Both L and U are triangular matrices and their determinants are equal to the product of diagonal elements, $\det(L) = 1$ and $\det(U) = -27$. Thus $\det(M) = -27$.

Problem 7 Every evening, Harald has either vanilla, chocolate, or strawberry ice cream for dessert. However, his choice is always influenced by his decision the day before.

- If Harald chose vanilla the prior evening, he has a 20% chance of again choosing vanilla, an 80% chance of choosing chocolate, and he never chooses strawberry.
- If Harald chose chocolate the prior evening, he has a 50% chance of choosing vanilla and a 20% chance of again choosing chocolate.
- If Harald chose strawberry the prior evening, he has a 30% chance of choosing vanilla, and a 70% chance of again choosing strawberry.

On a typical evening, what are the chances that Harald will choose each flavor of ice cream?

Solution We first wright down the transition matrix

$$P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

We are looking for a stationary vector of P that satisfies $P\mathbf{q} = \mathbf{q}$. For this we look at P_I and do Gauss elimination

$$P_I = \begin{bmatrix} -0.8 & 0.5 & 0.3 \\ 0.8 & -0.8 & 0 \\ 0 & 0.3 & -0.3 \end{bmatrix} \to \begin{bmatrix} -0.8 & 0.5 & 0.3 \\ 0 & -0.3 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\mathbf{q} = t[1,1,1]^T$. We choose $\mathbf{q}$ such that sum of the entries is equal to one, then $t = 1/3$ and $\mathbf{q} = [1/3, 1/3, 1/3]^T$. Thus the chances that Harald chooses each flavor is 1/3.

Problem 8 Assume that A is an invertible $n \times n$ matrix and vectors $\mathbf{v_1}$, $\mathbf{v_2}, \dots, \mathbf{v_n}$ form a basis for $\mathbb{R}^n$. Show that the vectors $A\mathbf{v_1}$, $A\mathbf{v_2}, \dots, A\mathbf{v_n}$ also form a basis for $\mathbb{R}^n$.

Solution Assume that $\mathbf{v_1}$, $\mathbf{v_2}, \dots, \mathbf{v_n}$ form a basis for $\mathbb{R}^n$. To show that n vectors $A\mathbf{v_1}$, $A\mathbf{v_2}, \dots, A\mathbf{v_n}$ also form a basis for $\mathbb{R}^n$ it is enough to show that those n vectors are linear independent. Assume that

$$c_1 A\mathbf{v_1} + c_2 A\mathbf{v_2} + ... + c_n A(v_n) = \mathbf{0}.$$

By the linearity of A it implies $A(c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}) = \mathbf{0}$. Since A is invertible we see that $c_1\mathbf{v_1} + ... + c_n\mathbf{v_n} = \mathbf{0}$. But by the assumption $\mathbf{v_1}$, $\mathbf{v_2}, \dots, \mathbf{v_n}$ are linearly independent (since they form a basis). Thus $c_1 = c_2 = ... = c_n$. This implies that $A\mathbf{v_1}$, $A\mathbf{v_2}, \dots, A\mathbf{v_n}$ are linearly independent as we wanted.