Department of Mathematical Sciences

Examination paper for **TMA4110/TMA4115 Matematikk 3**

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Examination time (from–to): 09:00-13:00

Permitted examination support material: C: Simple calculator (Casio fx-82ES PLUS, Citizen SR-270X or Citizen SR-270X College, Hewlett Packard HP30S), Rottmann: *Matematisk formelsamling*

Other information:

Give reasons for all answers, ensuring that it is clear how the answer has been reached. Each of the 12 problem parts has the same weight when grading.

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Checked by:

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Date Signature

Problem 1

a) Solve the equation $2\bar{z} - z = 1 - 6i$ and draw the solution on the complex plane.

Solution Let $z = x + iy$ then $\bar{z} = x - iy$ the equation can be written as $2x - 2iy - x - iy = 1 - 6i$. Comparing the real and imaginary parts, we get $x = 1$, $y = -2$. Thus the only solution is $\underline{z = 1 - 2i}$.

b) Find all solutions of the equation $z^5 = 1 + i$, writing your answer in polar form.

Solution We first write $1 + i$ in the polar form, $1 + i = \sqrt{2}e^{i\pi/4}$ then $z^5 = 1 + i$ has the following solutions:

$$\begin{aligned} z_0 &= \sqrt[10]{2}e^{i\pi/20}, \\ z_1 &= \sqrt[10]{2}e^{i\pi/20+2\pi/5} = \sqrt[10]{2}e^{9i\pi/20}, \\ z_2 &= \sqrt[10]{2}e^{i\pi/20+4\pi/5} = \sqrt[10]{2}e^{17i\pi/20}, \\ z_3 &= \sqrt[10]{2}e^{i\pi/20+6\pi/5} = \sqrt[10]{2}e^{5i\pi/4}, \\ z_4 &= \sqrt[10]{2}e^{i\pi/20+8\pi/5} = \sqrt[10]{2}e^{33i\pi/20}, \end{aligned}$$

Problem 2 Consider the equation

$$y'' - 9y = q(x).$$

a) Find the general solution of this equation when $q(x) = e^{3x}$.

Solution First to find the general solution to the homogeneous equation we solve the corresponding characteristic equation $\lambda^2 - 9\lambda = 0$, then $\lambda = \pm 3$ and the general solution is $y_h = C_1e^{3x} + C_2e^{-3x}$. To find a particular solution for the non-homogeneous equation with $q(x) = e^{3x}$ we use the method of undetermined coefficients. The modification rule should be applied since e^{3x} is a solution of the homogeneous equation and we look for a solution of the form $y_p = axe^{3x}$. We have $y_p' = ae^{3x} + 3axe^{3x}$ and $y_p'' = 6ae^{3x} + 9axe^{3x}$. Then $y_p'' - 9y_p = 6ae^{3x}$, to obtain $q(x) = e^{3x}$ we choose $a = 1/6$. Finally, the answer is $\underline{y(t) = 1/6xe^{3x} + C_1e^{3x} + c_2e^{-3x}}$.

b) Given that $y_1(x) = e^x \cos x$ is a solution, determine $q(x)$. For this $q(x)$ find the solution satisfying the initial conditions $y(0) = 0,\ y'(0) = 1$.

Solution First, if $y_1(x) = e^x \cos x$ is a solution, then

$$q(x) = y'' - 9y = e^x \cos x - 2e^x \sin x - e^x \cos x - 9e^x \cos x = \underline{-2e^x \sin x - 9e^x \cos x}.$$

The general solution of the equation is $y(x) = y_1(x) + y_h(x) = e^x \cos x + C_1 e^{3x} + C_2 e^{-3x}$. We find the values of C_1 and C_2 from the initial conditions. We have $y'(x) = e^x \cos x - e^x \sin x + 3C_1 e^{3x} - 3C_2 e^{-3x}$ and $0 = y(0) = 1 + C_1 + C_2$, $1 = y'(0) = 1 + 3C_1 - 3C_2$. Then $C_1 = C_2 = -0.5$. Thus

$$\underline{y(x) = e^x \cos x - 0.5e^{3x} - 0.5e^{-3x}}.$$

Problem 3 Find the general solution of the equation

$$y'' - 2y' + y = \frac{2e^t}{1+t^2}.$$

(Hint $\int (1+t^2)^{-1} dt = \arctan t$.)

Solution We will use variation of parameters. First, we solve the homogeneous equation, the characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, it has double root $\lambda = 1$ then two linearly independent solutions of the homogeneous equation are $y_1(t) = e^t$ and $y_2(t) = te^t$. We look for a particular solution in the form $y_p(t) = u_1 y_1 + u_2 y_2$. Solving the system

$$u_1' y_1 + u_2' y_2 = 0, \quad u_1' y_1' + u_2' y_2' = r(t) = \frac{2e^t}{1+t^2}$$

or using the formula, we obtain

$$u_1 = -\int \frac{y_2(t) r(t)}{W(t)} dt, \quad u_2 = \int \frac{y - 1(t) r(t)}{W(t)} dt,$$

where $W(t) = y_1 y_2' - y_1' y_2 = e^t(e^t + te^t) - e^t(te^t) = e^{2t}$ is the Wronski determinant of y_1 and y_2. Thus

$$u_1 = -\int \frac{2t}{1+t^2} dt = -\ln(1+t^2), \quad u_2 = \int \frac{2}{1+t^2} dt = 2 \arctan t.$$

Therefore

$$y_p(t) = u_1 y_1 + u_2 y_2 = -e^t \ln(1+t^2) + 2te^t \arctan t,$$

$$\underline{y(t) = y_p(t) + y_h(t) = -e^t \ln(1+t^2) + 2te^t \arctan t + (c_1 + tc_2)e^t}.$$

Problem 4 Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & t & -7 \\ 0 & 1 & 1-t \end{pmatrix}.$$

a) For which values of t is A invertible?

Solution The matrix is invertible if and only if $\det(A) \neq 0$. We compute the determinant using the top row expansion:

$$\det(A) = 2\begin{vmatrix} t & -7 \\ 1 & 1-t \end{vmatrix} - \begin{vmatrix} -1 & -7 \\ 0 & 1-t \end{vmatrix} = 2t(1-t) + 14 + (1-t) = -2t^2 + t + 15.$$

Solving the quadratic equation, we get $\det(A) = 0$ when $t = 3$ or $t = -2.5$. Thus A is invertible when $t \neq 3$ and $t \neq -2.5$.

b) Compute the inverse of A for $t = 0$.

Solution We compute the inverse by Gauss-Jordan elimination on the matrix $(A|I)$.

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -7 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2+1/2R_1]{1/2R_1} \left(\begin{array}{ccc|ccc} 1 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & -7 & 0.5 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1-R_2]{2R_2, R_3-2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 0 & -1 & 0 \\ 0 & 1 & -14 & 1 & 2 & 0 \\ 0 & 0 & 15 & -1 & -2 & 1 \end{array}\right) \xrightarrow[R_2+14R_3/15]{R_3/15, R_1-7R_3/15} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/15 & -1/15 & -7/15 \\ 0 & 1 & 0 & 1/15 & 2/15 & 14/15 \\ 0 & 0 & 1 & -1/15 & -2/15 & 1/15 \end{array}\right)$$

Thus $A^{-1} = \frac{1}{15}\begin{pmatrix} 7 & -1 & -7 \\ 1 & 2 & 14 \\ -1 & -2 & 1 \end{pmatrix}$.

Problem 5 Consider the following vectors in $\mathbb{R}^4$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 6 \\ -2 \\ -2 \\ -2 \end{pmatrix}.$$

a) Are these vectors linearly independent? Find a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution We consider matrix A with rows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we have

$$A = \begin{pmatrix} 1 & 3 & -2 & -2 \\ 0 & 2 & -1 & -1 \\ 6 & -2 & -2 & -2 \end{pmatrix}$$

Applying Gauss elimination, we get

$$\begin{pmatrix} 1 & 3 & -2 & -2 \\ 0 & 2 & -1 & -1 \\ 6 & -2 & -2 & -2 \end{pmatrix} \xrightarrow{R_3-6R_1} \begin{pmatrix} 1 & 3 & -2 & -2 \\ 0 & 2 & -1 & -1 \\ 0 & -20 & 10 & 10 \end{pmatrix} \xrightarrow{R_3+10R_2} \begin{pmatrix} 1 & 3 & -2 & -2 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors are linearly dependent and a basis for $Row(A)$ is a basis for the $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we may take $\mathbf{b}_1 = (1, 3, -2, -2)^T$, $\mathbf{b}_2 = (0, 2, -1, -1)^T$ then $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Alternative Solution Consider the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$,

$$\begin{bmatrix} 1 & 0 & 6 \\ 3 & 2 & -2 \\ -2 & -1 & -2 \\ -2 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & -20 \\ 0 & -1 & 10 \\ 0 & -1 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Only the first two columns contain pivot elements, thus vectors are linearly dependent and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b) Find a vector $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^4$ which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution We want to find $\mathbf{u}$ such that $\mathbf{v}_1 \cdot \mathbf{u} = 0$, $\mathbf{v}_2 \cdot \mathbf{u} = 0$ and $\mathbf{v}_3 \cdot \mathbf{u} = 0$. Thus $\mathbf{u}$ should satisfy $A\mathbf{u} = 0$, where A is the matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Using the Gauss elimination for A computed above, we see that the third and fourth variables are free and choosing for example $x_3 = 2, x_4 = 0$ we obtain $x_2 = 1/2(x_3 + x_4) = 1$ and $x_1 = 2(x_3 + x_4) - 3x_1 = 1$. Thus $\mathbf{u} = (1, 1, 2, 0)^T$ is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 6 Find the least squares line $y = mx + c$ that best fits the data points $\{(0, 0), (1, -2), (2, 1), (3, 4), (4, 2)\}$.

Solution We are looking for the least square solution of the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} c \\ m \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 4 \\ 2 \end{bmatrix}.$$

To find this solution we solve the normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ which is

$$\begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \end{bmatrix}.$$

The solution is $c = -1, m = 1$. Thus the best fitting line is $\underline{y = x - 1.}$

Problem 7 In the Spring in Trondheim, weather changes every hour. Each given hour, the weather can be sunny, rainy or snowy. The esteemed Institute for Baseless Estimations measured the following weather patterns.

- If the weather is sunny, there is 30 % chance that it will become rainy, and 30 % chance that it will become snowy the next hour.
- If it is rainy, there is 30 % chance that it will become sunny and 20 % chance that it will become snowy.
- If it is snowy, there is 20 % chance that it will stay snowy and 20 % chance that it will become rainy.

One day (after many hours of this pattern), you open your curtains. What is the most likely weather that you might see outside? (Give the probability for observing any of the three weather conditions.)

Solution The stochastic matrix which describes the hourly change of the weather is

$$A = \begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}$$

We want to find the stationary vector, it is a probability vector that satisfies $A\mathbf{v} = \mathbf{v}$. Thus we need to solve the system of linear equations with matrix $A - I$. The Gauss elimination gives

$$A_I = \begin{bmatrix} -0.6 & 0.3 & 0.6 \\ 0.3 & -0.5 & 0.2 \\ 0.3 & 0.2 & -0.8 \end{bmatrix} \to \begin{bmatrix} -1 & 0.5 & 1 \\ 0 & -0.35 & 0.5 \\ 0 & 0.35 & -0.5 \end{bmatrix}$$

Then solutions satisfy $x_3 = 0.7x_2,\ x_1 = x_3 + 0.5x_2 = 1.2x_2$. Then the stationary vector is

$$\mathbf{v} = [12/29, 10/29, 7/29] \sim [0.41, 0.34, 0.24].$$

The most likely weather outside is sunny.

Problem 8 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation that is one-to-one. Prove that if $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent vectors in $\mathbb{R}^n$ then the vectors $T(\mathbf{v}_1), ..., T(\mathbf{v}_k)$ are linearly independent in $\mathbb{R}^m$.

Solution Suppose that $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent.

To prove that $T(\mathbf{v}_1), ..., T(\mathbf{v}_k)$ are linearly independent let $c_1T(\mathbf{v}_1)+...+c_kT(\mathbf{v}_k) = \mathbf{0}$. We want to show that all constants $C_1, ..., c_k$ are zeros. By linearity of T we get $T(c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k) = \mathbf{0}$. It is given that T is a one-to-one linear transformation. Thus the only pre-image of the zero vector is the zero vector and $c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k = \mathbf{0}$. We know that $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent, therefore $c_1 = ... = c_k = 0$. Thus $T(\mathbf{v}_1), ..., T(\mathbf{v}_k)$ are linearly independent.