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TMA4115 - Calculus 3

Lecture 9, Feb 13

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Norwegian University of Science and Technology

Spring 2013

Review of last week's lecture

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- studied how to solve *systems of linear equations*,

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- studied how to solve *systems of linear equations*,
- introduced *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations,

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Review of last week's lecture

Last week we

- studied how to solve *systems of linear equations*,
- introduced *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations,
- introduced and studied *vectors*, *linear combinations* of vectors, subsets *spanned* by vectors, *vector equations*, the *product* of a matrix and a vector, *matrix equations*.

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Today's lecture

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- introduce and solve *homogeneous* and *nonhomegeneous* matrix equations,

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Today's lecture

Today we shall

- introduce and solve *homogeneous* and *nonhomegeneous* matrix equations,
- learn how to write solution sets in *parametric vector form*,
- look at applications of linear systems.

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Matrix equations

Theorem 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n = \mathbf{b},$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n \ldots \mathbf{b}].$$

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which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n \ldots \mathbf{b}].$$

Note that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combinations of the columns of A, that is, if and only if $\mathbf{b}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Homogeneous linear systems

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Homogeneous linear systems

- A system of linear equations is said to be *homogeneous* if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

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- Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely, $\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.

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- Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely, $\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.
- This zero solution is usually called the *trivial solution*.

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Homogeneous linear systems

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- Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely, $\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.
- This zero solution is usually called the *trivial solution*.
- The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example

Let us describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

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Example

Let us describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

We reduce the augmented matrix of the equation to its reduced echelon form.

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$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

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Example

Let us describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

We reduce the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

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Let us describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ where

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We reduce the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\rightarrow \begin{bmatrix} 1 & 5/3 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\to \begin{bmatrix} 1 & 5/3 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\to \begin{bmatrix} 1 & 5/3 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that

$$\begin{aligned} x_1 &= \frac{4}{3} x_3 \\ x_2 &= 0 \\ x_3 &\text{ is free} \end{aligned}$$

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Example (cont.)

The general solution of $A\mathbf{x} = \mathbf{0}$ is thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 is a free parameter.

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Parametric vector form

Parametric vector form

Whenever a solution set is written as

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors and $t_1, \ldots, t_p$ are parameters, we say that the solution is in *parametric vector form*.

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The solution set of a homogeneous equation

If the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix, can be written in parametric vector form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors and $t_1, \ldots, t_p$ are parameters, then the solution set is equal to Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

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Example

Let us write the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form where

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

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Example

Let us write the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form where

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

We reduce the augmented matrix of the equation to its reduced echelon form.

Example

Let us write the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form where

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

We reduce the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$$

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Example

Let us write the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form where

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

We reduce the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$$

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Example (cont.)

We see that

$$
\begin{aligned}
x_1 &= 2x_3 + 7x_4 \\
x_2 &= -2x_3 + 4x_4 \\
x_3 &\text{ is free} \\
x_4 &\text{ is free}
\end{aligned}
$$

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Example (cont.)

The general solution of $A\mathbf{x} = \mathbf{0}$ is thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 and x_4 are free parameters.

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Solutions of nonhomogeneous systems

Let A be an $m \times n$ matrix and that $\mathbf{b}$ is a vector in $\mathbb{R}^m$.

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Solutions of nonhomogeneous systems

Let A be an $m \times n$ matrix and that $\mathbf{b}$ is a vector in $\mathbb{R}^m$. Suppose that $\mathbf{x} = \mathbf{p}$ is a solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$, and that the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is given by the parametric form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the solution set of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is given by the parametric form

$$\mathbf{x} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p.$$

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Proof

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Proof

If $\mathbf{y} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$ where $t_1, t_2, \ldots, t_p$ are scalars,

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Proof

If $\mathbf{y} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$ where $t_1, t_2, \ldots, t_p$ are scalars, then

$$\begin{aligned} A\mathbf{y} &= A(\mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p) \\ &= A\mathbf{p} + t_1 A\mathbf{v}_1 + t_2 A\mathbf{v}_2 + \cdots + t_p A\mathbf{v}_p \\ &= \mathbf{b} + t_1\mathbf{0} + t_2\mathbf{0} + \ldots t_p\mathbf{0} = \mathbf{b}, \end{aligned}$$

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Proof

If $\mathbf{y} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$ where $t_1, t_2, \ldots, t_p$ are scalars, then

$$\begin{aligned} A\mathbf{y} &= A(\mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p) \\ &= A\mathbf{p} + t_1 A\mathbf{v}_1 + t_2 A\mathbf{v}_2 + \cdots + t_p A\mathbf{v}_p \\ &= \mathbf{b} + t_1\mathbf{0} + t_2\mathbf{0} + \ldots t_p\mathbf{0} = \mathbf{b}, \end{aligned}$$

so $\mathbf{x} = \mathbf{y} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

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Proof (cont.)

Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

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Proof (cont.)

Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{y} - \mathbf{p}) = A\mathbf{y} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

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Proof (cont.)

Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{y} - \mathbf{p}) = A\mathbf{y} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{x} = \mathbf{y} - \mathbf{p}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$.

Proof (cont.)

Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{y} - \mathbf{p}) = A\mathbf{y} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{x} = \mathbf{y} - \mathbf{p}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$. It follows that $\mathbf{y} - \mathbf{p} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$ for some scalars $t_1, t_2, \ldots, t_p$,

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Proof (cont.)

Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{y} - \mathbf{p}) = A\mathbf{y} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{x} = \mathbf{y} - \mathbf{p}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$. It follows that $\mathbf{y} - \mathbf{p} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$ for some scalars $t_1, t_2, \ldots, t_p$, and thus that $\mathbf{y} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$.

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Example

Let us write the general solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

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Let us write the general solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form where

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We reduce the augmented matrix of the equation to its reduced echelon form.

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$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

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Example

Let us write the general solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form where

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We reduce the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix}$$

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Example (cont.)

$$\to \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\to \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5/3 & -4/3 & 7/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\to \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5/3 & -4/3 & 7/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that

$$x_1 = \frac{4}{3}x_3 - 1$$

$$x_2 = 2$$

x_3 is free

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Example (cont.)

The general solution of $A\mathbf{x} = \mathbf{0}$ is thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

where x_3 is a free parameter.

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Writing a solution set of a consistent system in parametric vector form

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Writing a solution set of a consistent system in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.

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Writing a solution set of a consistent system in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.

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Writing a solution set of a consistent system in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.

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Writing a solution set of a consistent system in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

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Example

Let us write the general solution to the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 - 3x_3 &= 2 \\ -x_1 + x_2 \phantom{{}-3x_3} &= 0 \end{aligned}$$

in parametric vector form.

Example

Let us write the general solution to the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 - 3x_3 &= 2 \\ -x_1 + x_2 &= 0 \end{aligned}$$

in parametric vector form.
We reduce the augmented matrix of the system to its reduced echelon form.

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Example

Let us write the general solution to the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 - 3x_3 &= 2 \\ -x_1 + x_2 &= 0 \end{aligned}$$

in parametric vector form.
We reduce the augmented matrix of the system to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 2 & 1 & -3 & 2 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

Example

Let us write the general solution to the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 - 3x_3 &= 2 \\ -x_1 + x_2 &= 0 \end{aligned}$$

in parametric vector form.
We reduce the augmented matrix of the system to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 2 & 1 & -3 & 2 \\ -1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -3 & 3 & -2 \\ 0 & 3 & -3 & 2 \end{bmatrix}$$

Example (cont.)

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example (cont.)

$$\to \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 2/3 \\ 0 & 1 & -1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that

$$x_1 = x_3 + 2/3$$
$$x_2 = x_3 + 2/3$$
$$x_3 \text{ is free}$$

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Example (cont.)

The general solution of $A\mathbf{x} = \mathbf{0}$ is thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 + 2/3 \\ x_3 + 2/3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

where x_3 is a free parameter.

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Problem 3 from June 2005

Let $A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & -2 & -1 \\ 4 & -8 & 3 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ c \end{bmatrix}$ where c denotes an arbitrary real number.

1. Solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
2. For which values of c does the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ have a solution? Find the solution when the equation has a solution.

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Solution

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Solution

We reduce the augmented matrix $[A\ \mathbf{0}]$ to its reduced echelon form.

Solution

We reduce the augmented matrix $[A\ \mathbf{0}]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & 0 \end{bmatrix}$$

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Solution

We reduce the augmented matrix $[A\ \mathbf{0}]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & -5 & 5 & 0 \end{bmatrix}$$

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Solution

We reduce the augmented matrix $[A\ \mathbf{0}]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & -5 & 5 & 0 \end{bmatrix}$$

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$$\to \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\to \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

We see that

$$
\begin{aligned}
x_1 &= 2x_2 - x_4 \\
x_2 &= \text{ is free} \\
x_3 &= x_4 \\
x_4 &= \text{ is free}
\end{aligned}
$$

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Solution (cont.)

We see that

$$\begin{aligned} x_1 &= 2x_2 - x_4 \\ x_2 &= \text{ is free} \\ x_3 &= x_4 \\ x_4 &= \text{ is free} \end{aligned}$$

The general solution of $A\mathbf{x} = \mathbf{0}$ is thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are free parameters.

Solution (cont.)

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Solution (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

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Solution (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & c \end{bmatrix}$$

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Solution (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & c \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 4 & -4 & 12 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix}$$

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Solution (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & c \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 4 & -4 & 12 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix}$$

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Solution (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & c \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 4 & -4 & 12 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & c-1 \end{bmatrix}$$

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Solution (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ -3 & 6 & -2 & -1 & 0 \\ 4 & -8 & 3 & 1 & c \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 4 & -4 & 12 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -5 & 5 & c-16 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & c-1 \end{bmatrix}$$

We see that the equation has a solution if and only if $c = 1$.

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Solution (cont.)

If $c = 1$, then the reduced echelon form of $[A\ \mathbf{b}]$ is

$$\begin{bmatrix} 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

If $c = 1$, then the reduced echelon form of $[A\ \mathbf{b}]$ is

$$\begin{bmatrix} 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that

$$\begin{aligned} x_1 &= 2x_2 - x_4 - 2 \\ x_2 &= \text{ is free} \\ x_3 &= x_4 + 3 \\ x_4 &= \text{ is free} \end{aligned}$$

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Solution (cont.)

So when $c = 1$, then the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 - 2 \\ x_2 \\ x_4 + 3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

where x_2 and x_4 are free parameters.

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Example

Suppose an economy consist of the Coal, Electric, and Steel sectors, and the output of each sector is distributed among the various sectors as shown in the following table.

Coal	Electric	Steel	Purchased by:
0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

If possible, let us find equilibrium prices that make each sector's income match its expenditures.

Solution

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Solution

Let p_C be the price of the total annual output from the Coal sector, let p_E be the price of the total annual output from the Electric sector, and let p_S be the price of the total annual output from the Steel sector.

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Solution

Let p_C be the price of the total annual output from the Coal sector, let p_E be the price of the total annual output from the Electric sector, and let p_S be the price of the total annual output from the Steel sector.
To have equilibrium we mush have

$$\begin{aligned} p_C &= 0.4p_E + 0.6p_S \\ p_E &= 0.6p_C + 0.1p_E + 0.2p_S \\ p_S &= 0.4p_C + 0.5p_E + 0.2p_S \end{aligned}$$

Solution

which is equivalent to

$$\begin{aligned} p_C - 0.4p_E - 0.6p_S &= 0 \\ -0.6p_C + 0.9p_E - 0.2p_S &= 0 \\ -0.4p_C - 0.5p_E + 0.8p_S & \end{aligned}$$

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Solution

which is equivalent to

$$\begin{aligned} p_C - 0.4p_E - 0.6p_S &= 0 \\ -0.6p_C + 0.9p_E - 0.2p_S &= 0 \\ -0.4p_C - 0.5p_E + 0.8p_S & \end{aligned}$$

We write down the augmented matrix of the equation and row reduce it to reduced echelon form.

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix} \to$$

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Solution

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that we must have $p_C = 0.94 p_S$ and $p_E = 0.85 p_S$ in order to have equilibrium.

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Example

When propane burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O). Let us *balance* the chemical equation, that is find positive integers x_1 , x_2 , x_3, x_4 such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms are the same on the left match the corresponding numbers of atom on the right in the following equation.

$$x_1 C_3H_8 + x_2 O_2 \rightarrow x_3 CO_2 + x_4 H_2O$$

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Solution

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Solution

We must have that

$$\begin{aligned} 3x_1 &= x_3 \\ 8x_1 &= 2x_4 \\ 2x_2 &= 2x_3 + x_4 \end{aligned}$$

which is equivalent to

$$\begin{aligned} 3x_1 - x_3 &= 0 \\ 8x_1 - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

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Solution

We write down the augmented matrix of the equation and row reduce it to reduced echelon form.

$$\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 8/3 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 8/3 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{bmatrix}$$

Solution

We see that

$$\begin{aligned} x_1 &= 1/4x_4 \\ x_2 &= 5/4x_4 \\ x_3 &= 3/4x_4 \\ x_4 &\text{ is free} \end{aligned}$$

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Solution

We see that

$$\begin{aligned} x_1 &= 1/4x_4 \\ x_2 &= 5/4x_4 \\ x_3 &= 3/4x_4 \\ x_4 &\text{ is free} \end{aligned}$$

Since the coefficients in a chemical equation must be integers, we let $x_4 = 4$ and then $x_1 = 1$, $x_2 = 5$ and $x_3 = 3$.

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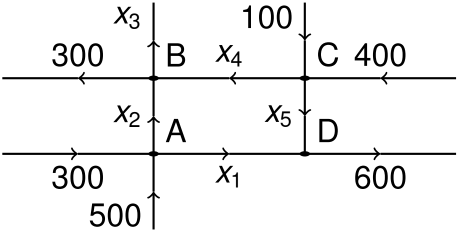
Example

The following network shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon.



Let us determine the general flow pattern for the network.

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Solution

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Solution

The number of cars that go into an intersection must be equal to the cars that leave the intersection. Also, the total number of cars that go into the network must be equal to the number of cars that leave the network.

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Solution

The number of cars that go into an intersection must be equal to the cars that leave the intersection. Also, the total number of cars that go into the network must be equal to the number of cars that leave the network. Thus we have

$$\begin{aligned} 300 + 500 &= x_1 + x_2 \\ x_2 + x_4 &= 300 + x_3 \\ 100 + 400 &= x_4 + x_5 \\ x_1 + x_5 &= 600 \\ 300 + 500 + 400 + 100 &= 300 + x_3 + 600 \end{aligned}$$

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Solution

which is equivalent to

$$
\begin{aligned}
x_1 + x_2 &= 800 \\
x_2 - x_3 + x_4 &= 300 \\
x_4 + x_5 &= 500 \\
x_1 + x_5 &= 600 \\
x_3 &= 400
\end{aligned}
$$

Solution

We write down the augmented matrix of the equation and row reduce it to reduced echelon form.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

Thus we have

$$\begin{aligned} x_1 &= 600 - x_5 \\ x_2 &= 200 + x_5 \\ x_3 &= 400 \\ x_4 &= 500 - x_5 \\ x_5 &\text{ is free} \end{aligned}$$

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Solution

Thus we have

$$
\begin{aligned}
x_1 &= 600 - x_5 \\
x_2 &= 200 + x_5 \\
x_3 &= 400 \\
x_4 &= 500 - x_5 \\
x_5 &\text{ is free}
\end{aligned}
$$

Since x_1, x_2, x_3, x_4 and x_5 denote number of cars, they must be nonnegative integers, so x_5 must be an integer between 0 and 500.

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Tomorrow's lecture

Tomorrow we shall introduce and study *linear dependence* and *linear independence* of vectors.
Section 1.7 in "Linear Algebras and Its Applications" (pages 55-62).

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