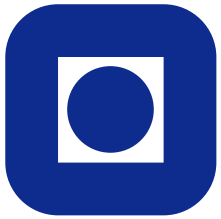

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TMA4115 - Calculus 3

Lecture 8, Feb 7

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday's lecture

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Yesterday's lecture

Yesterday we

Yesterday's lecture

Yesterday we

- studied how to solve *systems of linear equations*,

Yesterday's lecture

Yesterday we

- studied how to solve *systems of linear equations*,
- introduced *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

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Today's lecture

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Today we shall introduce and study

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- *vector equations*,

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Vectors

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- An example of a vector with two entries is

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where v_1 and v_2 are real numbers.

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- Let n be a positive integer. The set of all vectors with n real entries is denoted by $\mathbb{R}^n$.

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- The set of all vectors with n complex entries is denoted by $\mathbb{C}^n$.

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Vectors

- Two vectors in $\mathbb{R}^n$ are equal if and only if their corresponding entries are equal.

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- Given two vectors **u** and **v** in $\mathbb{R}^n$, their *sum* is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.

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Vectors

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- Given two vectors **u** and **v** in $\mathbb{R}^n$, their *sum* is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.
- Given a vector **u** and a real number c, the *scalar multiple* of **u** by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in **u** by c.

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Example

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Example

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

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Example

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Then

$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

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Then

$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}.$$

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Geometric description of $\mathbb{R}^2$ and $\mathbb{R}^3$

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Geometric description of $\mathbb{R}^2$ and $\mathbb{R}^3$

- We can identify a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ from $\mathbb{R}^2$ with the point (a, b) in the real plane.

Geometric description of $\mathbb{R}^2$ and $\mathbb{R}^3$

- We can identify a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ from $\mathbb{R}^2$ with the point (a, b) in the real plane.
- We can identify a column vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from $\mathbb{R}^3$ with the point (a, b, c) in the real 3-dimensional space.

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Algebraic properties of $\mathbb{R}^n$

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Algebraic properties of $\mathbb{R}^n$

The vector whose entries are all zero is called the *zero vector* and is denoted by **0**.

Algebraic properties of $\mathbb{R}^n$

The vector whose entries are all zero is called the *zero vector* and is denoted by $\mathbf{0}$. For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

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The vector whose entries are all zero is called the *zero vector* and is denoted by $\mathbf{0}$. For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

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4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

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5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

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7. $c(d\mathbf{u}) = (cd)\mathbf{u}$

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Algebraic properties of $\mathbb{R}^n$

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8. $1\mathbf{u} = \mathbf{u}$

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Linear combinations

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Linear combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$ the vector $\mathbf{w}$ defined by

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ with weights $c_1, c_2, \ldots, c_p$.

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is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ with weights $c_1, c_2, \ldots, c_p$. The weights in a linear combination can be any real numbers, including zero.

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Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

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Let us determine if $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

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Let us determine if $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}.$$

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Solution

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Solution

We must determine if there exists x_1 and x_2 such that

$$\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix}$$

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We must determine if there exists x_1 and x_2 such that

$$\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix}$$

that is

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

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Solution (cont.)

We reduce the augmented matrix of the system to an echelon form.

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Solution (cont.)

We reduce the augmented matrix of the system to an echelon form.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

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Solution (cont.)

We reduce the augmented matrix of the system to an echelon form.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix}$$

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We reduce the augmented matrix of the system to an echelon form.

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We see that the system is consistent, so $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

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Linear combinations

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2\ \ldots \mathbf{a}_p\ \mathbf{b}].$$

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A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2\ \ \ldots \mathbf{a}_p\ \mathbf{b}].$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ if and only if there exists a solution to the linear system corresponding to the matrix above.

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Subsets spanned by vectors

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Subsets spanned by vectors

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and is called the subset *spanned* (or *generated*) by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

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If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and is called the subset *spanned* (or *generated*) by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
That is, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

where $c_1, c_2, \ldots, c_p$ are scalars.

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A geometric description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

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A geometric description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

A geometric description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$.

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A geometric description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$. This set is the set of points on the line in $\mathbb{R}^2$ (or $\mathbb{R}^3$) through $\mathbf{v}$ and $\mathbf{0}$.

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A geometric description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$. This set is the set of points on the line in $\mathbb{R}^2$ (or $\mathbb{R}^3$) through $\mathbf{v}$ and $\mathbf{0}$.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$ with $\mathbf{u}$ not a multiple of $\mathbf{v}$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$.

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Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$

Asking whether a vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix

$$[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p\ \mathbf{b}]$$

has a solution.

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Note that Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

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Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

Let us determine if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.

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We reduce the augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}]$ to an echelon form.

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Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

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$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 9 \\ 3 & -3 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 9 \\ 3 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix}$$

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Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

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We see that the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ is inconsistent, so $\mathbf{b}$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

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Matrix-vector product

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Matrix-vector product

If A is an $m \times n$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{x}$ is in $\mathbb{R}^n$, then the *product of A and x*, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

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Matrix-vector product

If A is an $m \times n$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{x}$ is in $\mathbb{R}^n$, then the *product of A and x*, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$, and that the number of entries in $A\mathbf{x}$, if it is defined, is equal to the numbers of rows in A.

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Example

Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

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Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

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Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix}$$

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Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

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Example

Example

The linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

Example

The linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

is equivalent to the *vector equation*

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

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Example

The linear system

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and to the *matrix equation*

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

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Matrix equations

Theorem 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n \ldots \mathbf{b}].$$

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Matrix equations

Note that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combinations of the columns of A, that is, if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

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Example

Let $A \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

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Example

Let $A \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let us determine for which values of b_1, b_2 and b_3 the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

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Example (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

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$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

Example (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix}$$

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Example (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - b_2/2 + b_3 \end{bmatrix}$$

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Example (cont.)

We reduce the augmented matrix $[A\ \mathbf{b}]$ to an echelon form.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - b_2/2 + b_3 \end{bmatrix}$$

We see that the equation is consistent if and only if $b_1 - b_2/2 + b_3 = 0$.

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Matrix equations

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Matrix equations

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

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Proof

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Proof

Let **b** be in $\mathbb{R}^m$.

Proof

Let $\mathbf{b}$ be in $\mathbb{R}^m$. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A, and if and only if $\mathbf{b}$ is in the subset spanned by the columns of A.

Proof

Let $\mathbf{b}$ be in $\mathbb{R}^m$. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A, and if and only if $\mathbf{b}$ is in the subset spanned by the columns of A. The equivalence of (a), (b) and (c) follows from this.

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Proof

Let $\mathbf{b}$ be in $\mathbb{R}^m$. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A, and if and only if $\mathbf{b}$ is in the subset spanned by the columns of A. The equivalence of (a), (b) and (c) follows from this.
We will now show that (a) is true if (d) is true, and that (a) is false if (d) is false.

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Proof (cont.)

Let U be an echelon form of A.

Proof (cont.)

Let U be an echelon form of A. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[A\ \mathbf{b}]$ to an augmented matrix $[U\ \mathbf{d}]$ for some $\mathbf{d}$ in $\mathbb{R}^m$.

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Proof (cont.)

Let U be an echelon form of A. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[A\ \mathbf{b}]$ to an augmented matrix $[U\ \mathbf{d}]$ for some $\mathbf{d}$ in $\mathbb{R}^m$. If (d) is true (that is, A has a pivot position in every row), then each row of U contains a pivot position, and there can be no pivot in the last column of the augmented column $[U\ \mathbf{d}]$ for any $\mathbf{d}$ in $\mathbb{R}^m$.

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Proof (cont.)

Let U be an echelon form of A. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[A\ \mathbf{b}]$ to an augmented matrix $[U\ \mathbf{d}]$ for some $\mathbf{d}$ in $\mathbb{R}^m$. If (d) is true (that is, A has a pivot position in every row), then each row of U contains a pivot position, and there can be no pivot in the last column of the augmented column $[U\ \mathbf{d}]$ for any $\mathbf{d}$ in $\mathbb{R}^m$. It follows that there can be no pivot in the last column of the augmented column $[A\ \mathbf{b}]$ for any $\mathbf{b}$ in $\mathbb{R}^m$.

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Proof (cont.)

Let U be an echelon form of A. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[A\ \mathbf{b}]$ to an augmented matrix $[U\ \mathbf{d}]$ for some $\mathbf{d}$ in $\mathbb{R}^m$. If (d) is true (that is, A has a pivot position in every row), then each row of U contains a pivot position, and there can be no pivot in the last column of the augmented column $[U\ \mathbf{d}]$ for any $\mathbf{d}$ in $\mathbb{R}^m$. It follows that there can be no pivot in the last column of the augmented column $[A\ \mathbf{b}]$ for any $\mathbf{b}$ in $\mathbb{R}^m$. Thus the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true.

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Proof (cont.)

If (d) is false (that is, A does not have a pivot position in every row), then the last row of U is all zeros.

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Proof (cont.)

If (d) is false (that is, A does not have a pivot position in every row), then the last row of U is all zeros. Let $\mathbf{d}$ be any vector with an 1 in its last entry.

Proof (cont.)

If (d) is false (that is, A does not have a pivot position in every row), then the last row of U is all zeros. Let $\mathbf{d}$ be any vector with an 1 in its last entry. Then $[U\ \mathbf{d}]$ represents an inconsistent system.

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Proof (cont.)

If (d) is false (that is, A does not have a pivot position in every row), then the last row of U is all zeros. Let $\mathbf{d}$ be any vector with an 1 in its last entry. Then $[U\ \mathbf{d}]$ represents an inconsistent system. Since row operations are reversible, $[U\ \mathbf{d}]$ can be transformed into $[A\ \mathbf{b}]$ for some $\mathbf{b}$ in $\mathbb{R}^m$.

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Proof (cont.)

If (d) is false (that is, A does not have a pivot position in every row), then the last row of U is all zeros. Let $\mathbf{d}$ be any vector with an 1 in its last entry. Then $[U\ \mathbf{d}]$ represents an inconsistent system. Since row operations are reversible, $[U\ \mathbf{d}]$ can be transformed into $[A\ \mathbf{b}]$ for some $\mathbf{b}$ in $\mathbb{R}^m$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, from which it follows that (a) is false.

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Example

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Example

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Then

$$A\mathbf{x} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

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$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix}$$

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Let $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

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$$A\mathbf{x} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

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Then

$$A\mathbf{x} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

We see that the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

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Example

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Then

$$A\mathbf{x} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

We see that the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$. This is a general fact.

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Computations of $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example

Example

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & -5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

Example

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & -5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 2 \cdot (-4) + 0 \cdot 7 \\ -1 \cdot 2 + 0 \cdot (-4) + (-5) \cdot 7 \\ 1 \cdot 2 + 1 \cdot (-4) + 4 \cdot 7 \end{bmatrix}$$

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Example

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & -5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 2 \cdot (-4) + 0 \cdot 7 \\ -1 \cdot 2 + 0 \cdot (-4) + (-5) \cdot 7 \\ 1 \cdot 2 + 1 \cdot (-4) + 4 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -37 \\ 26 \end{bmatrix}$$

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Identities matrices

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Identities matrices

Let n be a positive integer. The matrix with 1s on the diagonal and 0s elsewhere is called an *identity matrix* and is denoted by I_n (or just I).

Identities matrices

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- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix I_3.

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Identities matrices

Let n be a positive integer. The matrix with 1s on the diagonal and 0s elsewhere is called an *identity matrix* and is denoted by I_n (or just I).

- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix I_3.
- $I_n\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$.

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Properties of the matrix-vector product

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Properties of the matrix-vector product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

Properties of the matrix-vector product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

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Properties of the matrix-vector product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Example

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$.

Let us determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$.

Let us determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

It follows from Theorem 4 that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$ if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ has a pivot position in every row.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$.

Let us determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

It follows from Theorem 4 that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$ if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ has a pivot position in every row.

So we reduce $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ to an echelon form.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$.

Let us determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

It follows from Theorem 4 that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$ if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ has a pivot position in every row.

So we reduce $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ to an echelon form.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix}$$

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$.

Let us determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

It follows from Theorem 4 that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$ if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ has a pivot position in every row.

So we reduce $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ to an echelon form.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 9 & -6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

We see that there is a pivot position in every row, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

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Problem 5 from August 2012

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

For which values of t does the equation $A\mathbf{x} = \mathbf{b}$ have none, one, or infinitely many solutions?

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Solution

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Solution

We start by reducing A to an echelon form.

Solution

We start by reducing A to an echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix}$$

Solution

We start by reducing A to an echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix}$$

Solution

We start by reducing A to an echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & t & t+1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix}$$

Solution

We start by reducing A to an echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & t & t+1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & t & t+1 \\ 0 & 0 & 0 & t^2-1 & t^2+t \end{bmatrix}$$

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Solution (cont.)

If $t = 1$, then $t^2 - 1 = 0$ and $t^2 + t \neq 0$, so the last column is a pivot column and the equation has no solution in that case.

Solution (cont.)

If $t = 1$, then $t^2 - 1 = 0$ and $t^2 + t \neq 0$, so the last column is a pivot column and the equation has no solution in that case.
If $t = -1$, then $t^2 - 1 = t^2 + t = 0$, so the last two columns are not pivot columns, and the equation is therefore consistent and has one free variable from which it follows that the equation has infinitely many solutions in that case.

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Solution (cont.)

If $t = 1$, then $t^2 - 1 = 0$ and $t^2 + t \neq 0$, so the last column is a pivot column and the equation has no solution in that case.
If $t = -1$, then $t^2 - 1 = t^2 + t = 0$, so the last two columns are not pivot columns, and the equation is therefore consistent and has one free variable from which it follows that the equation has infinitely many solutions in that case.
If $t \neq \pm 1$, then $t^2 - 1 \neq 0$, so all but the last column are pivot columns from which it follows that the equation has exactly one solution in that case.

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Plan for next week

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Plan for next week

Wednesday we shall

- introduce and solve *homogeneous* and *nonhomegeneous* matrix equations,
- learn how to write solution sets in *parametric vector form*,
- look at applications of linear systems.

Section 1.5-1.6 in “Linear Algebras and Its Applications” (pages 43-55).

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Plan for next week

Wednesday we shall

- introduce and solve *homogeneous* and *nonhomegeneous* matrix equations,
- learn how to write solution sets in *parametric vector form*,
- look at applications of linear systems.

Section 1.5-1.6 in “Linear Algebras and Its Applications” (pages 43-55).

Thursday we shall introduce and study

- define and study *linear independence* of vectors.

Section 1.7 in “Linear Algebras and Its Applications” (pages 55-62).

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