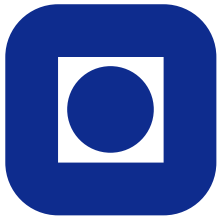

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TMA4115 - Calculus 3

Lecture 7, Feb 6

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of the previous lecture

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Review of the previous lecture

Last time we

Review of the previous lecture

Last time we

- looked at the method of *variation of parameters*,

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Review of the previous lecture

Last time we

- looked at the method of *variation of parameters*,
- studied *forced harmonic motions*.

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Today's lecture

Today's lecture

Today we shall

Today's lecture

Today we shall

- study how to solve *systems of linear equations*,

Today's lecture

Today we shall

- study how to solve *systems of linear equations*,
- introduce *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

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Linear equations

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Linear equations

A *linear equation* in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.

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Examples

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Examples

- The equations $4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$ are both linear.

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Examples

- The equations $4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$ are both linear.
- The equations $4x_1 - 5x_2 = x_1x_2$ and $x_2 = 2\sqrt{x_1} - 6$ are not linear.

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Systems of linear equations

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- The set of all possible solutions is called the *solution set* of the linear system.
- Two linear systems are called *equivalent* if they have the same solution set.

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Example

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Example

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= \ \ 3 \end{aligned}$$

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$(3, 2)$ is a solution of the system because

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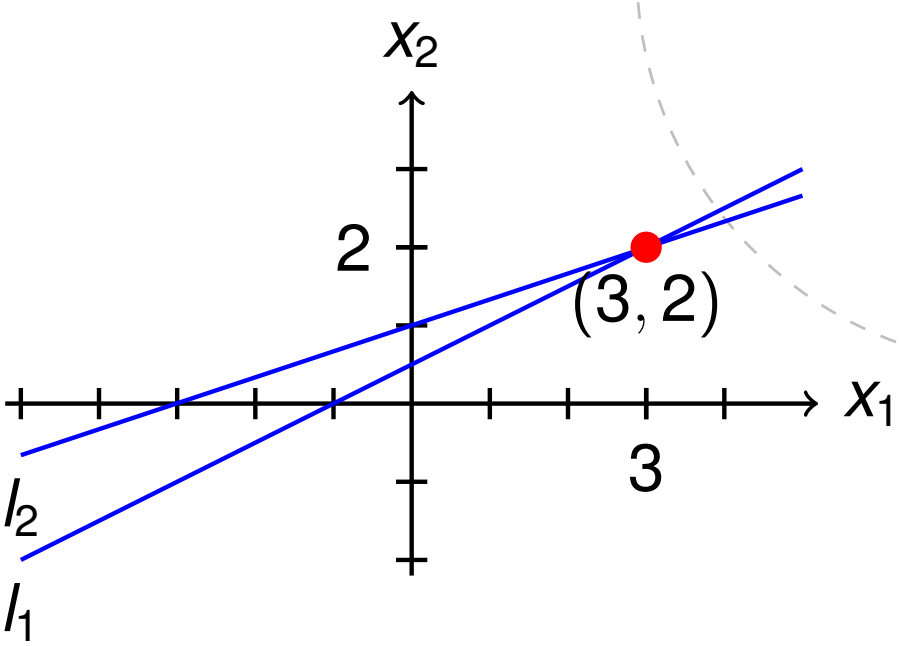

The graphs of the 2 equations are lines which we denote by l_1 and l_2. A pair of numbers (x_1, x_2) satisfies both equations if and only if the point (x_1, x_2) lies on both l_1 and l_2.

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$(3, 2)$ is a solution of the system because

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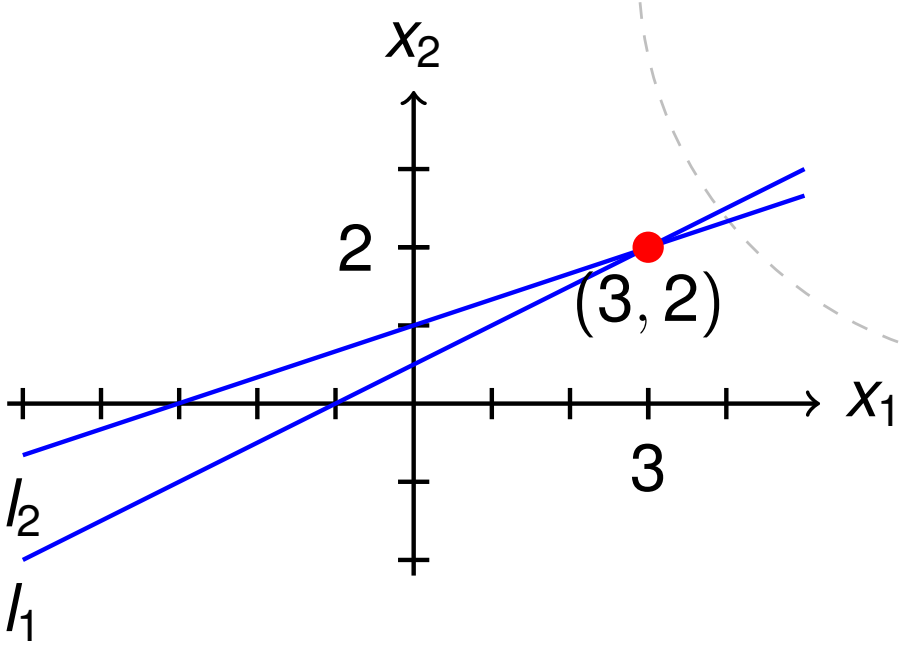
$(3, 2)$ is the only solution of the system, so $\{(3, 2)\}$ is the solution set of the system.



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- A system of linear equation is said to be *inconsistent* if it has no solution.

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Solving a linear system

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Solving a linear system

The basic strategy for solving a linear system is to replace the system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

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Example

Let us solve the linear system

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

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We add $4 \cdot$ [eq. 1] to [eq. 3].

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

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We multiply [eq. 2] by $1/2$.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

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Example

Let us solve the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

We add $4 \cdot$ [eq. 1] to [eq. 3].

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We add $3 \cdot$ [eq. 2] to [eq. 3].

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

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Example

We add $4 \cdot$ [eq. 3] to [eq. 2], and subtract [eq. 3] from [eq. 1].

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned}$$

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We add 2 · [eq. 2] to [eq. 1].

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned}$$

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So the solution set to the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

is $\{(29, 16, 3)\}$.

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Elementary row operations

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Elementary row operations

Elementary row operations include the following:

1. (Replacement) Replace one equation by the sum of itself and a multiple of another equation.

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Elementary row operations include the following:

1. (Replacement) Replace one equation by the sum of itself and a multiple of another equation.
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3. (Scaling) Multiply all coefficients in an equation by a nonzero constant.

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- If two linear systems are row equivalent, then they have the same solution set.
- Our strategy for solving a linear system is to use elementary row operations to transform the system into a system which is elementary to solve.

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Matrix notation

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- The essential information of a linear system can be recorded compactly in a rectangular array called a *matrix*.

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- For the following system of equations,

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the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the *coefficient matrix* of the system.

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Matrix notation

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Matrix size

- The size of a matrix tells how many rows and columns it has. If m and n are positive numbers, an $m \times n$ *matrix* is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)

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Matrix size

- The size of a matrix tells how many rows and columns it has. If m and n are positive numbers, an $m \times n$ *matrix* is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)
- If a system of linear equations consists of m equations and has n variables, then the coefficient matrix of the system is an $m \times n$ matrix, and the augmented matrix of the system is an $m \times (n + 1)$ matrix.

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- Our strategy for solving a linear system is to use elementary row operations to transform the augmented matrix of the system into a more elementary matrix.

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Example

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

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Example

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

We then add $-5/2$ times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

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Example

Next, we add $1/2$ times row 2 to row 3.

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This matrix is the augmented matrix of the system

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned}$$

which has no solutions.

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Example

It follows that the original system

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array}$$

does not have any solutions either.

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does not have any solutions either.
So it is inconsistent.

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Echelon form

Echelon form

- A *nonzero* row (or column) in a matrix is a row (or column) that contains at least one nonzero entry.

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Echelon form

- A *nonzero* row (or column) in a matrix is a row (or column) that contains at least one nonzero entry.
- A *leading entry* of a row is the leftmost nonzero entry in a nonzero row.

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- A *nonzero* row (or column) in a matrix is a row (or column) that contains at least one nonzero entry.
- A *leading entry* of a row is the leftmost nonzero entry in a nonzero row.
- A rectangular matrix is in *echelon form* (or *row echelon form*) if it has the following three properties:

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  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

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Reduced echelon form

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Reduced echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or *reduced row echelon form*):

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Reduced echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or *reduced row echelon form*):

4. The leading entry in each nonzero row is 1.

Reduced echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or *reduced row echelon form*):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Echelon form

- An *echelon matrix* (respectively, *reduced echelon matrix*) is one that is in echelon form (respectively, reduced echelon form).

Echelon form

- An *echelon matrix* (respectively, *reduced echelon matrix*) is one that is in echelon form (respectively, reduced echelon form).

The following matrices are in echelon form. The leading entries ■ may have any nonzero value; the starred entries $*$ may have any value (including 0).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Reduced echelon form

The following matrices are in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

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Row reduction

Row reduction

Any nonzero matrix may be *row reduced* (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

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Any nonzero matrix may be *row reduced* (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

Thm. 1: Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

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Thm. 1: Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix U, we call U *an echelon form* (or *row echelon form*) of A; if U is in reduced echelon form, we call U *the reduced echelon form* of A.

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Pivot position

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- Since the reduced echelon form of a matrix A is unique, the leading entries are always in the same positions in any echelon form of A.

Pivot position

- Since the reduced echelon form of a matrix A is unique, the leading entries are always in the same positions in any echelon form of A.
- A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position.

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Row reduction algorithm

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Row reduction algorithm

Let us apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Row reduction algorithm

Let us apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

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Row reduction algorithm

Step 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

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Row reduction algorithm

Step 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

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Row reduction algorithm

Step 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

We interchange rows 1 and 3.

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$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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Row reduction algorithm

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3

Use row replacement operations to create zeros in all positions below the pivot.

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Row reduction algorithm

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3

Use row replacement operations to create zeros in all positions below the pivot.

We add -1 times row 1 to row 2.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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Row reduction algorithm

Step 4

Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1—3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

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Row reduction algorithm

Step 4

Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1—3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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Row reduction algorithm

For step 3, we add $-3/2$ times row 2 to row 3.

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Row reduction algorithm

For step 3, we add $-3/2$ times row 2 to row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row reduction algorithm

For step 3, we add $-3/2$ times row 2 to row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

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Row reduction algorithm

For step 3, we add $-3/2$ times row 2 to row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row. Steps 1—3 require no work for this submatrix, and we have reached an echelon form of the full matrix.

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Row reduction algorithm

We perform one more step to obtain the reduced echelon form.

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Row reduction algorithm

We perform one more step to obtain the reduced echelon form.

Step 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

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Row reduction algorithm

The rightmost pivot is in row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row reduction algorithm

The rightmost pivot is in row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

We create zeros above it by adding -2 times row 3 to row 2, and -6 times row 3 to row 1.

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Row reduction algorithm

The rightmost pivot is in row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

We create zeros above it by adding -2 times row 3 to row 2, and -6 times row 3 to row 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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Row reduction algorithm

The next pivot is in row 2.

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Row reduction algorithm

The next pivot is in row 2. Scale this row, dividing by the pivot.

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Row reduction algorithm

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row reduction algorithm

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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Row reduction algorithm

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row reduction algorithm

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is the reduced echelon form of the original matrix.

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Row reduction algorithm

The combination of steps 1—4 is called the *forward phase* of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the *backward phase*.

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Row reduction algorithm

The combination of steps 1—4 is called the *forward phase* of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the *backward phase*. The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

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Solutions of linear systems

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Solutions of linear systems

Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solutions of linear systems

Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 variables because the augmented matrix has four columns.

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Solutions of linear systems

The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

Solutions of linear systems

The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called *basic variables*. The other variable, x_3, is called a *free variable*.

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Solutions of linear systems

- Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

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Solutions of linear systems

- Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.
- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

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Solutions of linear systems

In the example

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

we solve the first and second equations for x_1 and x_2 (we ignore the third equation; it offers no restriction on the variables).

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Solutions of linear systems

We then get

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
x_3 is free

Solutions of linear systems

We then get

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
$$x_3 \text{ is free}$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas above determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is (1,4,0); when $x_3 = 1$, the solution is (6,3,1).

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Parametric description of solution sets

- The description

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
$$x_3 \text{ is free}$$

  is a parametric description of solutions sets in which the free variables act as parameters.

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Parametric description of solution sets

- The description

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
$$x_3 \text{ is free}$$

  is a parametric description of solutions sets in which the free variables act as parameters.
- Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

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Parametric description of solution sets

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

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Parametric description of solution sets

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. As an example, let us consider the linear system

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

whose augmented matrix is the matrix

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we considered before.

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Parametric description of solution sets

If we add 5 times equation 2 to equation 1, we obtain the following equivalent system.

$$\begin{aligned} x_1 + 5x_2 &= 21 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

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Parametric description of solution sets

If we add 5 times equation 2 to equation 1, we obtain the following equivalent system.

$$\begin{aligned} x_1 + 5x_2 &= 21 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.

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Inconsistent systems

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Inconsistent systems

When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

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Existence and uniqueness theorem

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Existence and uniqueness theorem

Theorem 2: Existence and uniqueness theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 \dots 0 & b \end{bmatrix}$$

with b nonzero.

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Existence and uniqueness theorem

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If a linear system is consistent, then the solution set contains either

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Existence and uniqueness theorem

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with b nonzero.
If a linear system is consistent, then the solution set contains either

1. a unique solution, when there are no free variables, or

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Existence and uniqueness theorem

Theorem 2: Existence and uniqueness theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 \dots 0 & b \end{bmatrix}$$

with b nonzero.
If a linear system is consistent, then the solution set contains either

1. a unique solution, when there are no free variables, or
2. infinitely many solutions, when there is at least on free variable.

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Solving a linear system

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Solving a linear system

To use row reduction to solve a linear system, we do the following:

Solving a linear system

To use row reduction to solve a linear system, we do the following:

1. Write the augmented matrix of the system.

Solving a linear system

To use row reduction to solve a linear system, we do the following:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

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Solving a linear system

To use row reduction to solve a linear system, we do the following:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.

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Solving a linear system

To use row reduction to solve a linear system, we do the following:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.

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Solving a linear system

To use row reduction to solve a linear system, we do the following:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

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Example 1

Let us solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 6x_3 & = & -8 \\ & & x_2 & + & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

Example 1

Let us solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 6x_3 & = & -8 \\ & & x_2 & + & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

Solution:

Example 1

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$$\begin{array}{rcrcrcr} 2x_1 & & & - & 6x_3 & = & -8 \\ & & x_2 & + & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

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Example 1

Let us solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 6x_3 & = & -8 \\ & & x_2 & + & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

We row reduce the augmented matrix to an echelon form.

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix}$$

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Since the last matrix doesn't have a pivot position in the last column, the system is consistent.

Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Since the last matrix doesn't have a pivot position in the last column, the system is consistent. We therefore continue and find the reduced echelon form.

Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is the reduced echelon form of the system.

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Example 1

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is the reduced echelon form of the system. We see that the system has a unique solution which is $x_1 = 2$, $x_2 = -1$, $x_3 = 2$.

Example 2

Let us solve the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 3x_3 & = & 7 \\ & - & 2x_2 & + & 4x_3 & = & 0 \\ 3x_1 & + & 2x_2 & - & 7x_3 & = & 21 \end{array}$$

Example 2

Let us solve the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 3x_3 & = & 7 \\ & - & 2x_2 & + & 4x_3 & = & 0 \\ 3x_1 & + & 2x_2 & - & 7x_3 & = & 21 \end{array}$$

Solution:

Example 2

Let us solve the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 3x_3 & = & 7 \\ & - & 2x_2 & + & 4x_3 & = & 0 \\ 3x_1 & + & 2x_2 & - & 7x_3 & = & 21 \end{array}$$

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix}$$

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Example 2

Let us solve the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 3x_3 & = & 7 \\ & - & 2x_2 & + & 4x_3 & = & 0 \\ 3x_1 & + & 2x_2 & - & 7x_3 & = & 21 \end{array}$$

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix}$$

We row reduce the augmented matrix to an echelon form.

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Example 2

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix}$$

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Example 2

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -9 & 15 & -7 \end{bmatrix}$$

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Example 2

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -9 & 15 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

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Example 2

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -9 & 15 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Since the last matrix has a pivot position in the last column, the system is inconsistent. The system therefore has no solution.

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Example 3

Let us solve the linear system

$$\begin{array}{rcrcrcrcr} 2x_1 & & & & & - & 4x_4 & = & -10 \\ & & 3x_2 & + & 3x_3 & & & = & 0 \\ -3x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 5 \end{array}$$

Example 3

Let us solve the linear system

$$\begin{array}{rcrcrcrcr} 2x_1 & & & & & - & 4x_4 & = & -10 \\ & & 3x_2 & + & 3x_3 & & & = & 0 \\ -3x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 5 \end{array}$$

Solution:

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Example 3

Let us solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & & & & - & 4x_4 & = & -10 \\ & & 3x_2 & + & 3x_3 & & = & 0 \\ -3x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 5 \end{array}$$

Solution: The augmented matrix of the system is

$$\left[\begin{array}{rrrrr} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{array}\right]$$

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Example 3

Let us solve the linear system

$$\begin{array}{rcrcrcrcr} 2x_1 & & & & & - & 4x_4 & = & -10 \\ & & 3x_2 & + & 3x_3 & & & = & 0 \\ -3x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 5 \end{array}$$

Solution: The augmented matrix of the system is

$$\left[\begin{array}{rrrrr} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{array}\right]$$

We row reduce the augmented matrix to an echelon form.

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Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 2 & 3 & -5 & -10 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 2 & 3 & -5 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 2 & 3 & -5 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

Since the last matrix doesn't have a pivot position in the last column, the system is consistent.

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Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 2 & 3 & -5 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

Since the last matrix doesn't have a pivot position in the last column, the system is consistent. We therefore continue and find the reduced echelon form.

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Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 0 & 15 & 30 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

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Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 0 & 15 & 30 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 0 & 15 & 30 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

The last matrix is the reduced echelon form of the system.

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Example 3

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 0 & 15 & 30 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix}$$

The last matrix is the reduced echelon form of the system. Since the first 3 columns are pivot columns and column 4 is not, x_1, x_2 and x_3 are basic variables, while x_4 is a free variable.

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Example 3

The reduced echelon form corresponds to the system

$$x_1 - 2x_4 = -5$$
$$x_2 + 5x_4 = 10$$
$$x_3 - 5x_4 = -10$$

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Example 3

The reduced echelon form corresponds to the system

$$x_1 - 2x_4 = -5$$
$$x_2 + 5x_4 = 10$$
$$x_3 - 5x_4 = -10$$

We solve for the basic variables to obtain the general solution

$$x_1 = -5 + 2x_4$$
$$x_2 = 10 - 5x_4$$
$$x_3 = -10 + 5x_4$$
$$x_4 \text{ is free}$$

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Problem 7 from August 2004

1. Solve the systems of equation

$$\begin{aligned} x + 2y &= 1 \\ 3x + 7y + 2z &= 2 \\ 2x + 3y - 3z &= 0. \end{aligned}$$

Problem 7 from August 2004

2. Decide for which values of the parameters a and b the system of equations

$$\begin{aligned} x + 2y &= 1 \\ 3x + 7y + 2z &= b \\ 2x + 3y - az &= 0 \end{aligned}$$

has

1. exactly one solution
2. infinitely two solutions
3. infinitely many solutions
4. no solutions.

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Solution

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Solution

The augmented matrix of the system

$$\begin{aligned} x + 2y &= 1 \\ 3x + 7y + 2z &= 2 \\ 2x + 3y - 3z &= 0. \end{aligned}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix}$$

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Solution

The augmented matrix of the system

$$\begin{aligned} x + 2y &= 1 \\ 3x + 7y + 2z &= 2 \\ 2x + 3y - 3z &= 0. \end{aligned}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix}$$

We row reduce the augmented matrix to an echelon form.

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

Since the last matrix doesn't have a pivot position in the last column, the system is consistent.

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

Since the last matrix doesn't have a pivot position in the last column, the system is consistent. We therefore continue and find the reduced echelon form.

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The last matrix is the reduced echelon form of the system.

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The last matrix is the reduced echelon form of the system. We see that the system has a unique solution which is $x_1 = 15$, $x_2 = -7$, $x_3 = 3$.

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Solution

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Solution

The augmented matrix of the system

$$\begin{aligned} x + 2y &= 1 \\ 3x + 7y + 2z &= 2 \\ 2x + 3y - 3z &= 0. \end{aligned}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix}$$

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Solution

The augmented matrix of the system

$$\begin{aligned} x + 2y &= 1 \\ 3x + 7y + 2z &= 2 \\ 2x + 3y - 3z &= 0. \end{aligned}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix}$$

We row reduce the augmented matrix to an echelon form.

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & -1 & -a & -2 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & -1 & -a & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & 0 & 2-a & b-5 \end{bmatrix}$$

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & -1 & -a & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & 0 & 2-a & b-5 \end{bmatrix}$$

If $a = 2$ and $b \neq -5$, then the last column is a pivot column and the system has no solutions.

Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & -1 & -a & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & 0 & 2-a & b-5 \end{bmatrix}$$

If $a = 2$ and $b \neq -5$, then the last column is a pivot column and the system has no solutions.

If $a = 2$ and $b = 5$, then the last two columns are not pivot columns, and the system has infinitely many solutions.

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Solution

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 2 & b \\ 2 & 3 & -a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & -1 & -a & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & b-3 \\ 0 & 0 & 2-a & b-5 \end{bmatrix}$$

If $a = 2$ and $b \neq -5$, then the last column is a pivot column and the system has no solutions.
If $a = 2$ and $b = 5$, then the last two columns are not pivot columns, and the system has infinitely many solutions.
If $a \neq 2$, then the first 3 columns are pivot columns and the last column is not, so the system has exactly one solution.

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Plan for tomorrow

Tomorrow we shall introduce and study

- *vectors*,
- *linear combinations of vectors*,
- subsets *spanned* by vectors,
- *vector equations*,
- the *product* of a matrix and a vector,
- *matrix equations*.

Section 1.3-1.4 in “Linear Algebras and Its Applications” (pages 24-42).

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