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TMA4115 - Calculus 3

Lecture 6, Jan 31

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Spring 2013

Review of yesterday's lecture

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Review of yesterday's lecture

Yesterday we

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Review of yesterday's lecture

Yesterday we

- studied harmonic motions,

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Review of yesterday's lecture

Yesterday we

- studied harmonic motions,
- studied solutions of second-order linear inhomogeneous differential equations,

Review of yesterday's lecture

Yesterday we

- studied harmonic motions,
- studied solutions of second-order linear inhomogeneous differential equations,
- looked at the method of undetermined coefficients.

Today's lecture

Today we shall

Today's lecture

Today we shall

- look at *variation of parameters*,

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Today's lecture

Today we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

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General solutions to inhomogeneous equations

If y_p is a particular solution to the inhomogeneous equation $y'' + py' + qy = f$ and y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$, then the general solution to the inhomogeneous equation $y'' + py' + qy = f$ is

$$y = y_p + c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are constants.

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The method of undetermined coefficients

Consider the inhomogeneous second-order linear differential equation

$$y'' + py' + qy = f.$$

If the function f has a form that is replicated under differentiation, then look for a solution with the same general form as f.

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Problem 2 from June 2012

1. Find a particular solution of $y'' - 4y' + y = te^t + t$.
2. Find the solution of $y'' - 4y' + y = te^t + t$, where $y'(0) = y(0) = 0$.

Solution

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Solution

Let $y(t) = ate^t + be^t + ct + d$.

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Solution

Let $y(t) = ate^t + be^t + ct + d$. Then
$y'(t) = ae^t + ate^t + be^t + c = ate^t + (a + b)e^t + c$,

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Solution

Let $y(t) = ate^t + be^t + ct + d$. Then
$y'(t) = ae^t + ate^t + be^t + c = ate^t + (a + b)e^t + c$,
$y''(t) = ae^t + ate^t + (a + b)e^t = ate^t + (2a + b)e^t$,

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Solution

Let $y(t) = ate^t + be^t + ct + d$. Then
$y'(t) = ae^t + ate^t + be^t + c = ate^t + (a+b)e^t + c$,
$y''(t) = ae^t + ate^t + (a+b)e^t = ate^t + (2a+b)e^t$, and

$$\begin{aligned} y''(t) - 4y'(t) + y(t) &= ate^t + (2a+b)e^t \\ &\quad - 4(ate^t + (a+b)e^t + c) \\ &\quad + ate^t + be^t + ct + d \\ &= -2ate^t - (2a+2b)e^t + ct + d - 4c \end{aligned}$$

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Solution

Let $y(t) = ate^t + be^t + ct + d$. Then
$y'(t) = ae^t + ate^t + be^t + c = ate^t + (a+b)e^t + c$,
$y''(t) = ae^t + ate^t + (a+b)e^t = ate^t + (2a+b)e^t$, and

$$\begin{aligned} y''(t) - 4y'(t) + y(t) = {} & ate^t + (2a+b)e^t \\ & - 4(ate^t + (a+b)e^t + c) \\ & + ate^t + be^t + ct + d \\ = {} & -2ate^t - (2a+2b)e^t + ct + d - 4c \end{aligned}$$

so $y(t) = ate^t + be^t + ct + d$ is a solution of
$y'' - 4y' + y = te^t + t$ if and only if $a = -\frac{1}{2}$, $b = -a = \frac{1}{2}$, $c = 1$
and $d = c4 = 4$.

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Solution

Let $y(t) = ate^t + be^t + ct + d$. Then
$y'(t) = ae^t + ate^t + be^t + c = ate^t + (a+b)e^t + c$,
$y''(t) = ae^t + ate^t + (a+b)e^t = ate^t + (2a+b)e^t$, and

$$\begin{aligned} y''(t) - 4y'(t) + y(t) &= ate^t + (2a+b)e^t \\ &\quad - 4(ate^t + (a+b)e^t + c) \\ &\quad + ate^t + be^t + ct + d \\ &= -2ate^t - (2a+2b)e^t + ct + d - 4c \end{aligned}$$

so $y(t) = ate^t + be^t + ct + d$ is a solution of
$y'' - 4y' + y = te^t + t$ if and only if $a = -\frac{1}{2}$, $b = -a = \frac{1}{2}$, $c = 1$
and $d = c4 = 4$. So $y(t) = \frac{-1}{2}te^t + \frac{1}{2}e^t + t + 4$ is a particular
solution of $y'' - 4y' + y = te^t + t$.

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Solution

To find the solution of $y'' - 4y' + y = te^t + t$ where $y'(0) = y(0) = 0$, we will first find the general solution of $y'' - 4y' + y = te^t + t$.

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Solution

To find the solution of $y'' - 4y' + y = te^t + t$ where $y'(0) = y(0) = 0$, we will first find the general solution of $y'' - 4y' + y = te^t + t$. To do that, we will first find the general solution of $y'' - 4y' + y = 0$.

Solution

To find the solution of $y'' - 4y' + y = te^t + t$ where $y'(0) = y(0) = 0$, we will first find the general solution of $y'' - 4y' + y = te^t + t$. To do that, we will first find the general solution of $y'' - 4y' + y = 0$. The characteristic polynomial of $y'' - 4y' + y = 0$ is $\lambda^2 - 4\lambda + 1$,

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Solution

To find the solution of $y'' - 4y' + y = te^t + t$ where $y'(0) = y(0) = 0$, we will first find the general solution of $y'' - 4y' + y = te^t + t$. To do that, we will first find the general solution of $y'' - 4y' + y = 0$. The characteristic polynomial of $y'' - 4y' + y = 0$ is $\lambda^2 - 4\lambda + 1$, and the characteristic roots are $\lambda = \dfrac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$,

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Solution

To find the solution of $y'' - 4y' + y = te^t + t$ where $y'(0) = y(0) = 0$, we will first find the general solution of $y'' - 4y' + y = te^t + t$. To do that, we will first find the general solution of $y'' - 4y' + y = 0$. The characteristic polynomial of $y'' - 4y' + y = 0$ is $\lambda^2 - 4\lambda + 1$, and the characteristic roots are $\lambda = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$, so the general solution of $y'' - 4y' + y = 0$ is $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t}$.

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Solution

To find the solution of $y'' - 4y' + y = te^t + t$ where $y'(0) = y(0) = 0$, we will first find the general solution of $y'' - 4y' + y = te^t + t$. To do that, we will first find the general solution of $y'' - 4y' + y = 0$. The characteristic polynomial of $y'' - 4y' + y = 0$ is $\lambda^2 - 4\lambda + 1$, and the characteristic roots are $\lambda = \dfrac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$, so the general solution of $y'' - 4y' + y = 0$ is $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t}$. It follows that $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + \frac{1}{2} e^t + t + 4$ is the general solution of $y'' - 4y' + y = te^t + t$.

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Solution

If $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + \frac{1}{2} e^t + t + 4$, then
$y'(t) = c_1(2+\sqrt{3})e^{(2+\sqrt{3})t} + c_2(2-\sqrt{3})e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + 1$,

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Solution

If $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + \frac{1}{2} e^t + t + 4$, then
$y'(t) = c_1(2+\sqrt{3})e^{(2+\sqrt{3})t} + c_2(2-\sqrt{3})e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + 1$,
$y(0) = c_1 + c_2 + \frac{9}{2}$,

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Solution

If $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + \frac{1}{2} e^t + t + 4$, then
$y'(t) = c_1(2+\sqrt{3})e^{(2+\sqrt{3})t} + c_2(2-\sqrt{3})e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + 1$,
$y(0) = c_1 + c_2 + \frac{9}{2}$, and $y'(0) = c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) + 1$,

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Solution

If $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + \frac{1}{2} e^t + t + 4$, then
$y'(t) = c_1(2+\sqrt{3})e^{(2+\sqrt{3})t} + c_2(2-\sqrt{3})e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + 1$,
$y(0) = c_1 + c_2 + \frac{9}{2}$, and $y'(0) = c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) + 1$,
so $y'(0) = y(0) = 0$ if and only if $c_1 + c_2 = -\frac{9}{2}$ and
$c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) = -1$.

Solution

If $y(t) = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + \frac{1}{2} e^t + t + 4$, then
$y'(t) = c_1(2+\sqrt{3})e^{(2+\sqrt{3})t} + c_2(2-\sqrt{3})e^{(2-\sqrt{3})t} + \frac{-1}{2} te^t + 1$,
$y(0) = c_1 + c_2 + \frac{9}{2}$, and $y'(0) = c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) + 1$,
so $y'(0) = y(0) = 0$ if and only if $c_1 + c_2 = -\frac{9}{2}$ and
$c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) = -1$.
The solution of the linear system

$$c_1 + c_2 = -\frac{9}{2}$$
$$c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) = -1$$

is $c_1 = \frac{4}{\sqrt{3}} - \frac{9}{4}$ and $c_2 = \frac{-4}{\sqrt{3}} - \frac{9}{4}$.

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Solution

Thus

$$y(t) = \left(\frac{4}{\sqrt{3}} - \frac{9}{4}\right) e^{(2+\sqrt{3})t} + \left(\frac{-4}{\sqrt{3}} - \frac{9}{4}\right) e^{(2-\sqrt{3})t} + \frac{-1}{2} te^{t} + \frac{1}{2} e^{t} + t + 4$$

is a solution of $y'' - 4y' + y = te^t + t$ which satisfies that $y'(0) = y(0) = 0$.

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Forced harmonic motion

We will now apply the technique of undetermined coefficients to analyze harmonic motion with an external sinusoidal forcing term.

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Forced harmonic motion

We will now apply the technique of undetermined coefficients to analyze harmonic motion with an external sinusoidal forcing term.
The equation we need to solve is

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

Forced undamped harmonic motion

Let us first assume that $c = 0$.

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Forced undamped harmonic motion

Let us first assume that $c = 0$. Then the equation becomes

$$y'' + \omega_0^2 y = A\cos(\omega t).$$

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Forced undamped harmonic motion

Let us first assume that $c = 0$. Then the equation becomes

$$y'' + \omega_0^2 y = A\cos(\omega t).$$

The general solution to the homogeneous solution $y'' + \omega_0^2 y = 0$ is $y_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$.

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Forced undamped harmonic motion

Let us first assume that $c = 0$. Then the equation becomes

$$y'' + \omega_0^2 y = A\cos(\omega t).$$

The general solution to the homogeneous solution $y'' + \omega_0^2 y = 0$ is $y_h = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$.
Let us find a particular solution by using the method of undetermined coefficients.

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Forced undamped harmonic motion

Let us first assume that $c = 0$. Then the equation becomes

$$y'' + \omega_0^2 y = A\cos(\omega t).$$

The general solution to the homogeneous solution $y'' + \omega_0^2 y = 0$ is $y_h = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$.
Let us find a particular solution by using the method of undetermined coefficients. We will first look at the case where $\omega \neq \omega_0$, and then at the case where $\omega = \omega_0$.

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The case $\omega \neq \omega_0$

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

$$y_p''(t) + \omega_0^2 y_p(t)$$

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

$$\begin{aligned} y_p''(t) + \omega_0^2 y_p(t) = &-a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) \\ &+ a\omega_0^2\cos(\omega t) + b\omega_0^2\sin(\omega t) \end{aligned}$$

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

$$\begin{aligned} y_p''(t) + \omega_0^2 y_p(t) &= -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) \\ &\quad + a\omega_0^2\cos(\omega t) + b\omega_0^2\sin(\omega t) \\ &= a(\omega_0^2 - \omega^2)\cos(\omega t) + b(\omega_0^2 - \omega^2)\sin(\omega t) \end{aligned}$$

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

$$\begin{aligned} y_p''(t) + \omega_0^2 y_p(t) &= -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) \\ &\quad + a\omega_0^2\cos(\omega t) + b\omega_0^2\sin(\omega t) \\ &= a(\omega_0^2 - \omega^2)\cos(\omega t) + b(\omega_0^2 - \omega^2)\sin(\omega t) \end{aligned}$$

so y_p is a particular solution if and only if $a = \frac{A}{\omega_0^2 - \omega^2}$ and $b = 0$.

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

$$\begin{aligned} y_p''(t) + \omega_0^2 y_p(t) &= -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) \\ &\qquad + a\omega_0^2\cos(\omega t) + b\omega_0^2\sin(\omega t) \\ &= a(\omega_0^2 - \omega^2)\cos(\omega t) + b(\omega_0^2 - \omega^2)\sin(\omega t) \end{aligned}$$

so y_p is a particular solution if and only if $a = \frac{A}{\omega_0^2 - \omega^2}$ and $b = 0$. Thus $y_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t)$ is a particular solution,

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The case $\omega \neq \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ of the form $y_p = a\cos(\omega t) + b\sin(\omega t)$.

$$
\begin{aligned}
y_p''(t) + \omega_0^2 y_p(t) &= -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) \\
&\quad + a\omega_0^2\cos(\omega t) + b\omega_0^2\sin(\omega t) \\
&= a(\omega_0^2 - \omega^2)\cos(\omega t) + b(\omega_0^2 - \omega^2)\sin(\omega t)
\end{aligned}
$$

so y_p is a particular solution if and only if $a = \frac{A}{\omega_0^2 - \omega^2}$ and $b = 0$. Thus $y_p(t) = \frac{A}{\omega_0^2-\omega^2}\cos(\omega t)$ is a particular solution, and the general solution is
$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{\omega_0^2-\omega^2}\cos(\omega t)$.

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The case $\omega \neq \omega_0$

The general solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ is
$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t)$.

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The case $\omega \neq \omega_0$

The general solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ is $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t)$.
Let us look at the solution where the motion starts at equilibrium.

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The case $\omega \neq \omega_0$

The general solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ is $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t)$.
Let us look at the solution where the motion starts at equilibrium. This means that $y(0) = y'(0) = 0$.

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The case $\omega \neq \omega_0$

The general solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ is $y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t)$.

Let us look at the solution where the motion starts at equilibrium. This means that $y(0) = y'(0) = 0$. We then have that $0 = y(0) = c_1 + \frac{A}{\omega_0^2 - \omega^2}$ and $0 = y'(0) = c_2\omega_0$,

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The case $\omega \neq \omega_0$

The general solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ is $y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{\omega_0^2-\omega^2}\cos(\omega t)$.
Let us look at the solution where the motion starts at equilibrium. This means that $y(0) = y'(0) = 0$. We then have that $0 = y(0) = c_1 + \frac{A}{\omega_0^2-\omega^2}$ and $0 = y'(0) = c_2\omega_0$, so
$y(t) = \frac{A}{\omega_0^2-\omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.

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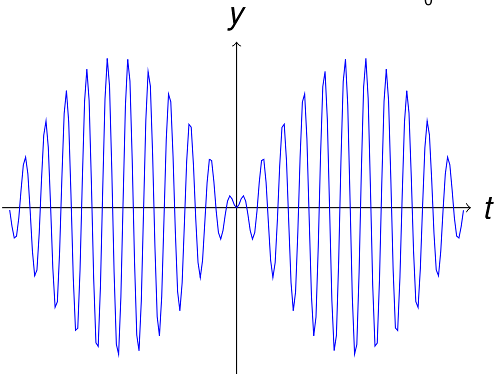
The case $\omega \neq \omega_0$

$$y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$$



The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.

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The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.
Let $\overline{\omega} = (\omega_0 + \omega)/2$ and $\delta = (\omega_0 - \omega)/2$.

The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.
Let $\overline{\omega} = (\omega_0 + \omega)/2$ and $\delta = (\omega_0 - \omega)/2$. $\overline{\omega}$ is called the *mean frequency*, and δ is called the *half difference*.

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The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.

Let $\overline{\omega} = (\omega_0 + \omega)/2$ and $\delta = (\omega_0 - \omega)/2$. $\overline{\omega}$ is called the *mean frequency*, and δ is called the *half difference*. We then have that

$$y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$$

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The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.

Let $\overline{\omega} = (\omega_0 + \omega)/2$ and $\delta = (\omega_0 - \omega)/2$. $\overline{\omega}$ is called the *mean frequency*, and δ is called the *half difference*. We then have that

$$
\begin{aligned}
y(t) &= \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t)) \\
&= \frac{A}{4\overline{\omega}\delta}(\cos((\overline{\omega} - \delta)t) - \cos((\overline{\omega} + \delta)t))
\end{aligned}
$$

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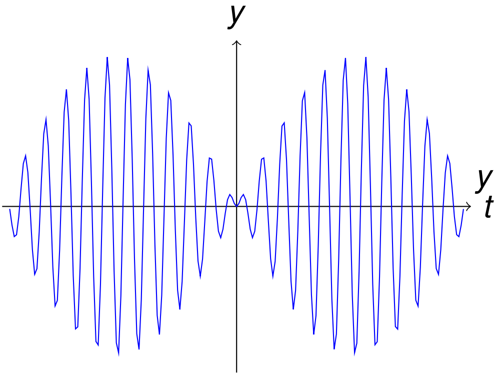
The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.

Let $\overline{\omega} = (\omega_0 + \omega)/2$ and $\delta = (\omega_0 - \omega)/2$. $\overline{\omega}$ is called the *mean frequency*, and δ is called the *half difference*. We then have that

$$\begin{aligned} y(t) &= \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t)) \\ &= \frac{A}{4\overline{\omega}\delta}(\cos((\overline{\omega} - \delta)t) - \cos((\overline{\omega} + \delta)t)) \\ &= \frac{A\sin(\delta t)}{2\overline{\omega}\delta} \sin(\overline{\omega} t). \end{aligned}$$

The case $\omega \neq \omega_0$



$$y(t) = \frac{A\sin(\delta t)}{2\overline{\omega}\delta}\sin(\overline{\omega}t)$$

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The case $\omega = \omega_0$

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The case $\omega = \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega_0 t)$.

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The case $\omega = \omega_0$

We will look for a solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega_0 t)$. Since $A\cos(\omega_0 t)$ is a solution to the homogeneous equation $y'' + 2cy' + \omega_0^2 y = 0$, we will look for a solution of the form $y_p = t(a\cos(\omega_0 t) + b\sin(\omega_0 t))$.

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$$
\begin{aligned}
y_p''(t) + \omega_0^2 y_p(t) = 2\omega_0(-a\sin(\omega_0 t) + b\cos(\omega_0 t)) \\
+ t\omega_0^2(-a\cos(\omega_0 t) - b\sin(\omega_0 t)) \\
+ \omega_0^2 t(a\cos(\omega_0 t) + b\sin(\omega_0 t))
\end{aligned}
$$

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So y_p is a particular solution if and only if $a = 0$ and $b = \frac{A}{2\omega_0}$.

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The case $\omega = \omega_0$

Thus $y_p(t) = \frac{A}{2\omega_0} t \sin(\omega_0 t)$ is a particular solution,

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The case $\omega = \omega_0$

Thus $y_p(t) = \frac{A}{2\omega_0} t \sin(\omega_0 t)$ is a particular solution, and the general solution is
$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{A}{2\omega_0} t \sin(\omega_0 t)$.

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Let us look at the solution where the motion starts at equilibrium.

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Let us look at the solution where the motion starts at equilibrium. This means that $y(0) = y'(0) = 0$. We then have that $0 = y(0) = c_1$ and $0 = y'(0) = c_2\omega_0$, so
$y(t) = \frac{A}{2\omega_0} t \sin(\omega_0 t)$.

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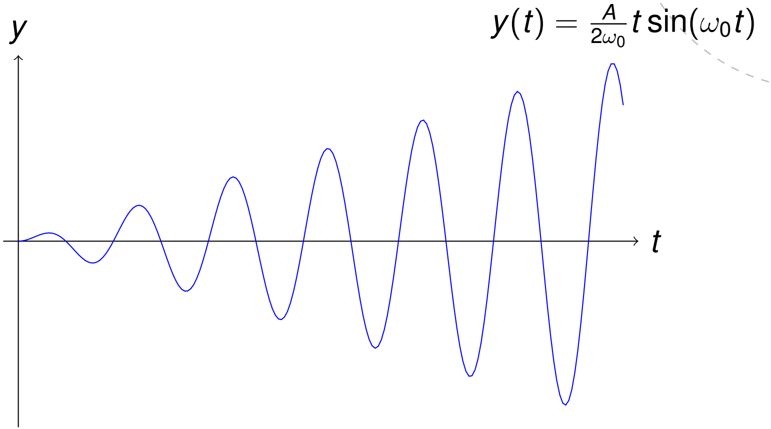
The case $\omega = \omega_0$



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The forced damped harmonic motion

The forced damped harmonic motion

If we add a damping term to the system we get the equation

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

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If we add a damping term to the system we get the equation

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Let us assume that $c < \omega_0$.

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If we add a damping term to the system we get the equation

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Let us assume that $c < \omega_0$. Then the general solution to the homogeneous equation $y'' + 2cy' + \omega_0^2 y = 0$ is

$y_h(t) = e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t))$ where $\eta = \sqrt{\omega_0^2 - c^2}$.

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To find a particular solution we will use the technique of undetermined coefficients and the complex method. This means that we will be looking for a solution $z(t) = ae^{i\omega t}$ to the equation $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$.

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The forced damped harmonic motion

If $z(t) = ae^{i\omega t}$,

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The forced damped harmonic motion

If $z(t) = ae^{i\omega t}$, then

$z''(t) + 2cz(t)' + \omega_0^2 z(t) =$

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The forced damped harmonic motion

If $z(t) = ae^{i\omega t}$, then

$$z''(t)+2cz(t)'+\omega_0^2 z(t) = ((i\omega)^2+2c(i\omega)+\omega_0^2)ae^{i\omega t}$$

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The forced damped harmonic motion

If $z(t) = ae^{i\omega t}$, then

$$z''(t)+2cz(t)'+\omega_0^2 z(t) = ((i\omega)^2+2c(i\omega)+\omega_0^2)ae^{i\omega t} = P(i\omega)ae^{i\omega t}$$

where $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ is the characteristic polynomial.

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where $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ is the characteristic polynomial. Thus $z(t) = \frac{A}{P(i\omega)}e^{i\omega t} = H(i\omega)Ae^{i\omega t}$ is a solution to $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$ where $H(i\omega) = \frac{1}{P(i\omega)}$.

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Let $R = |P(i\omega)|$ and $\phi = \mathrm{Arg}(P(i\omega))$.

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If $z(t) = ae^{i\omega t}$, then

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where $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ is the characteristic polynomial. Thus $z(t) = \frac{A}{P(i\omega)}e^{i\omega t} = H(i\omega)Ae^{i\omega t}$ is a solution to $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$ where $H(i\omega) = \frac{1}{P(i\omega)}$. The function $H(i\omega)$ is called the *transfer function*.
Let $R = |P(i\omega)|$ and $\phi = \text{Arg}(P(i\omega))$. Then $P(i\omega) = Re^{i\phi}$ and $H(i\omega) = \frac{1}{R}e^{-i\phi}$.

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The forced damped harmonic motion

So $z(t) = H(i\omega)Ae^{i\omega t} = \frac{1}{R}e^{-i\phi}Ae^{i\omega t} = \frac{A}{R}e^{i(\omega t-\phi)}$,

The forced damped harmonic motion

So $z(t) = H(i\omega)Ae^{i\omega t} = \frac{1}{R}e^{-i\phi}Ae^{i\omega t} = \frac{A}{R}e^{i(\omega t-\phi)}$, and $y_p(t) = \mathrm{Re}(z(t)) = \frac{A}{R}\cos(\omega t - \phi)$ is a particular solution to $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$.

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The general solution to

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

is $y(t) = e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t)) + \frac{A}{R}\cos(\omega t - \phi)$.

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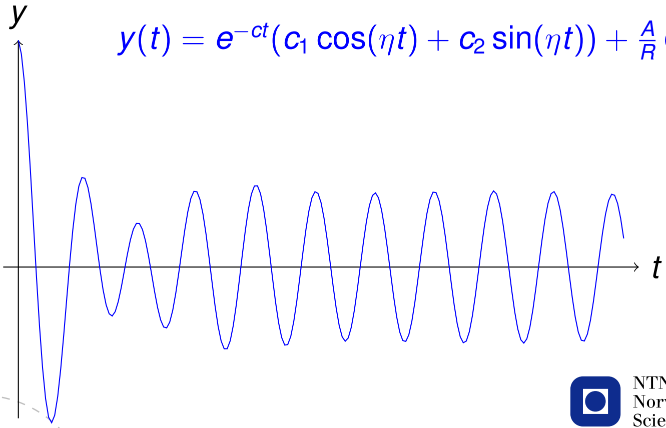
The forced damped harmonic motion

$$y(t) = e^{-ct}(c_1 \cos(\eta t) + c_2 \sin(\eta t)) + \frac{A}{R} \cos(\omega t - \phi)$$



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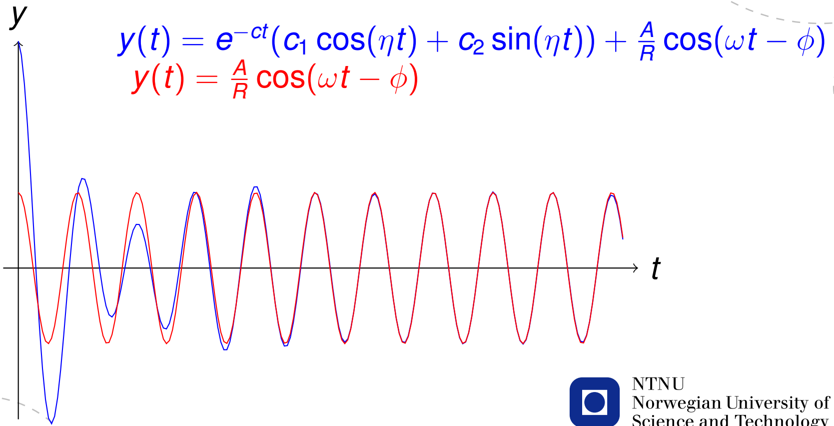

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Steady-state and transient terms

The general solution to

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The term $e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t))$ is called the *transition term*, and the term $\frac{A}{R}\cos(\omega t - \phi)$ is called the *steady-state* term.

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Variation of parameters

We are looking for a particular solution to an inhomogeneous second-order linear differential equation

$$y'' + p(t)y' + q(t)y = f(t).$$

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Suppose that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$.

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Variation of parameters

We are looking for a particular solution to an inhomogeneous second-order linear differential equation

$$y'' + p(t)y' + q(t)y = f(t).$$

Suppose that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$. The idea behind the *variation of parameters* method is to look for a particular solution of the form $y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ where v_1 and v_2 are unknown functions.

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Example

Let us find a particular solution to the equation

$$y'' + y = \tan(t).$$

Example

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The functions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ form a fundamental set of solutions of the equation $y'' + y = 0$,

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The functions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ form a fundamental set of solutions of the equation $y'' + y = 0$, so we let

$$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = v_1(t)\cos(t) + v_2(t)\sin(t).$$

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Then

$y'(t) = v_1'(t)\cos(t) - v_1(t)\sin(t) + v_2'(t)\sin(t) + v_2(t)\cos(t).$

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Then

$y'(t) = v_1'(t)\cos(t) - v_1(t)\sin(t) + v_2'(t)\sin(t) + v_2(t)\cos(t)$.

Let us assume that $v_1'(t)\cos(t) + v_2'(t)\sin(t) = 0$.

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The functions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ form a fundamental set of solutions of the equation $y'' + y = 0$, so we let

$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$.

Then

$y'(t) = v_1'(t)\cos(t) - v_1(t)\sin(t) + v_2'(t)\sin(t) + v_2(t)\cos(t)$.

Let us assume that $v_1'(t)\cos(t) + v_2'(t)\sin(t) = 0$. Then

$y'(t) = -v_1(t)\sin(t) + v_2(t)\cos(t)$,

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Example

and

$$\begin{aligned} y''(t) + y(t) &= -v_1'(t)\sin(t) - v_1(t)\cos(t) \\ &\quad + v_2'(t)\cos(t) - v_2(t)\sin(t) \\ &\quad + v_1(t)\cos(t) + v_2(t)\sin(t) \\ &= -v_1'(t)\sin(t) + v_2'(t)\cos(t). \end{aligned}$$

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So $y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ is a solution to $y'' + y = \tan(t)$ if $v_1'(t)\cos(t) + v_2'(t)\sin(t) = 0$ and $-v_1(t)\sin(t) + v_2(t)\cos(t) = \tan(t)$.

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Example

The solution of the linear system

$$
\begin{aligned}
v_1'(t)\cos(t) + v_2'(t)\sin(t) &= 0 \\
-v_1(t)\sin(t) + v_2(t)\cos(t) &= \tan(t)
\end{aligned}
$$

is

$$
\begin{aligned}
v_1'(t) &= \frac{-\tan(t)\sin(t)}{\cos^2(t)+\sin^2(t)} = -\tan(t)\sin(t) = \frac{-\sin^2(t)}{\cos(t)} \\
v_2'(t) &= \frac{\tan(t)\cos(t)}{\cos^2(t)+\sin^2(t)} = \tan(t)\cos(t) = \sin(t).
\end{aligned}
$$

Example

So if we let

$$v_1(t) = \int \frac{-\sin^2(t)}{\cos(t)} dt = \int \frac{\cos^2(t) - 1}{\cos(t)} dt$$

$$= \int \cos(t) - \frac{1}{\cos(t)} = \sin(t) - \ln|\sec(t) + \tan(t)|$$

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Example

So if we let

$$v_1(t) = \int \frac{-\sin^2(t)}{\cos(t)}dt = \int \frac{\cos^2(t)-1}{\cos(t)}dt$$
$$= \int \cos(t) - \frac{1}{\cos(t)} = \sin(t) - \ln|\sec(t) + \tan(t)|$$

and

$$v_2(t) = \int \sin(t)dt = -\cos(t)$$

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Example

then

$$\begin{aligned} y(t) &= v_1(t)y_1(t) + v_2(t)y_2(t) \\ &= \big(\sin(t) - \ln|\sec(t) + \tan(t)|\big)\cos(t) - \cos(t)\sin(t) \end{aligned}$$

is a particular solution to the equation $y'' + y = \tan(t)$.

Variation of parameters

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Variation of parameters

To find a particular solution to $y'' + py' + qy = f$ using the method of variation of parameters we follow these steps.

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Variation of parameters

To find a particular solution to $y'' + py' + qy = f$ using the method of variation of parameters we follow these steps.

1. Find a fundamental set of solutions y_1, y_2 to the homogeneous equation $y'' + py' + qy = 0$.

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To find a particular solution to $y'' + py' + qy = f$ using the method of variation of parameters we follow these steps.

1. Find a fundamental set of solutions y_1, y_2 to the homogeneous equation $y'' + py' + qy = 0$.
2. Let $y_p = v_1 y_1 + v_2 y_2$ where v_1 and v_2 are functions to be determined.

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2. Let $y_p = v_1 y_1 + v_2 y_2$ where v_1 and v_2 are functions to be determined.
3. Find v_1' and v_2' such that

$$
\begin{aligned}
v_1' y_1 + v_2' y_2 &= 0 \\
v_1' y_1' + v_2' y_2' &= f.
\end{aligned}
$$

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3. Find v_1' and v_2' such that

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = f.$$

4. Let $v_1(t) = \int v_1'(t)\, dt$ and $v_2(t) = \int v_2'(t)\, dt$.

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$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= f. \end{aligned}$$
4. Let $v_1(t) = \int v_1'(t)\,dt$ and $v_2(t) = \int v_2'(t)\,dt$.
5. Substitute v_1 and v_2 into $y_p = v_1 y_1 + v_2 y_2$.

Variation of parameters

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Variation of parameters

If

$$v_1(t) = \int \frac{-y_2(t)f(t)}{W(t)}\,dt$$

and

$$v_2(t) = \int \frac{y_1(t)f(t)}{W(t)}\,dt$$

where $W(t)$ is the Wronskian of y_1 and y_2,

Variation of parameters

If

$$v_1(t) = \int \frac{-y_2(t)f(t)}{W(t)}\,dt$$

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$$v_2(t) = \int \frac{y_1(t)f(t)}{W(t)}\,dt$$

where $W(t)$ is the Wronskian of y_1 and y_2, then

$$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

is a solution of $y'' + py' + qy = f$.

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Problem 2 August 2012

Find the solution of $y'' - 2y' + y = \frac{e^x}{x}$ for $x > 0$ which satisfies $y(1) = y'(1) = 0$.

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Solution

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Solution

We will us variation of parameters to find a particular solution of $y'' - 2y' + y = \frac{e^x}{x}$, so we first have to find a fundamental set of solutions to $y'' - 2y + y = 0$.

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Solution

We will us variation of parameters to find a particular solution of $y'' - 2y' + y = \frac{e^x}{x}$, so we first have to find a fundamental set of solutions to $y'' - 2y + y = 0$.
The characteristic polynomial of $y'' - 2y + y = 0$ is
$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$,

Solution

We will us variation of parameters to find a particular solution of $y'' - 2y' + y = \frac{e^x}{x}$, so we first have to find a fundamental set of solutions to $y'' - 2y + y = 0$.
The characteristic polynomial of $y'' - 2y + y = 0$ is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so $y_1(x) = e^x$ and $y_2(x) = xe^x$ form a fundamental set of solutions to $y'' - 2y + y = 0$.

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Solution

We will us variation of parameters to find a particular solution of $y'' - 2y' + y = \frac{e^x}{x}$, so we first have to find a fundamental set of solutions to $y'' - 2y + y = 0$.
The characteristic polynomial of $y'' - 2y + y = 0$ is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so $y_1(x) = e^x$ and $y_2(x) = xe^x$ form a fundamental set of solutions to $y'' - 2y + y = 0$.
The Wronskian of y_1 and y_2 is

$$\begin{aligned} W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^x(e^x + xe^x) - e^x xe^x = e^{2x}. \end{aligned}$$

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Solution

It follows that if we let

$$v_1(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-xe^x e^x/x}{e^{2x}}dx = \int -dx = -x$$

Solution

It follows that if we let

$$v_1(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-xe^x e^x/x}{e^{2x}}dx = \int -dx = -x$$

and

$$v_2(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{e^x e^x/x}{e^{2x}}dx = \int \frac{1}{x}dx = \ln(x),$$

Solution

It follows that if we let

$$v_1(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-xe^x e^x/x}{e^{2x}}dx = \int -dx = -x$$

and

$$v_2(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{e^x e^x/x}{e^{2x}}dx = \int \frac{1}{x}dx = \ln(x),$$

then

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x) = -xe^x + xe^x\ln(x)$$

is a particular solution of $y'' - 2y' + y = \frac{e^x}{x}$.

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Solution

It follows that the general solution of $y'' - 2y' + y = \frac{e^x}{x}$ is

$$y(x) = c_1 e^x + c_2 x e^x - x e^x + x e^x \ln(x).$$

Solution

It follows that the general solution of $y'' - 2y' + y = \frac{e^x}{x}$ is

$$y(x) = c_1 e^x + c_2 x e^x - x e^x + x e^x \ln(x).$$

If

$$\begin{aligned} y(x) &= c_1 e^x + c_2 x e^x - x e^x + x e^x \ln(x) \\ &= c_1 e^x + (c_2 - 1 + \ln(x)) x e^x, \end{aligned}$$

then

$$\begin{aligned} y'(x) = c_1 e^x + \frac{1}{x} x e^x + (c_2 - 1 + \ln(x)) e^x \\ + (c_2 - 1 + \ln(x)) x e^x, \end{aligned}$$

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Solution

$$y(1) = c_1 e + (c_2 - 1)e = (c_1 + c_2 - 1)e$$

and

$$y'(1) = c_1 e + e + (c_2 - 1)e + (c_2 - 1)e = (c_1 + 2c_2 - 1)e,$$

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Solution

$$y(1) = c_1 e + (c_2 - 1)e = (c_1 + c_2 - 1)e$$

and

$$y'(1) = c_1 e + e + (c_2 - 1)e + (c_2 - 1)e = (c_1 + 2c_2 - 1)e,$$

so $y(x) = c_1 e^x + (c_2 - 1 + \ln(x))xe^x$ satisfies
$y(1) = y'(1) = 0$ if and only if $c_1 + c_2 = 1$ and $c_1 + 2c_2 = 1$.

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Solution

$$y(1) = c_1 e + (c_2 - 1)e = (c_1 + c_2 - 1)e$$

and

$$y'(1) = c_1 e + e + (c_2 - 1)e + (c_2 - 1)e = (c_1 + 2c_2 - 1)e,$$

so $y(x) = c_1 e^x + (c_2 - 1 + \ln(x))xe^x$ satisfies $y(1) = y'(1) = 0$ if and only if $c_1 + c_2 = 1$ and $c_1 + 2c_2 = 1$. The solution of the linear system

$$c_1 + c_2 = 1$$
$$c_1 + 2c_2 = 1$$

is $c_1 = 1$ and $c_2 = 0$.

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Solution

So $y(x) = e^x - xe^x + \ln(x)xe^x$ is a solution of $y'' - 2y' + y = \frac{e^x}{x}$ on $(0, \infty)$ which satisfies $y(1) = y'(1) = 0$.

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Problem 2 December 2010

1. The motion of a mechanical system is described by the differential equation

$$y'' + 6y' + 18y = 0.$$

   Determine whether the motion is under-damped, is over-damped or is critically damped. Find a particular solution $y(t)$ that satisfies the initial conditions $y(0) = 0$, $y'(0) = 0.6$.

2. Find the steady-state solution of the equation

$$y'' + 6y' + 18y = 45 \cos 3t.$$

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Solution

Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$,

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Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$, and the characteristic roots are
$$\lambda = \frac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i.$$

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Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$, and the characteristic roots are $\lambda = \dfrac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i$. It follows that the system is under-damped and that the general solution of $y'' + 6y' + 18y = 0$ is $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$.

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Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$, and the characteristic roots are $\lambda = \dfrac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i$. It follows that the system is under-damped and that the general solution of $y'' + 6y' + 18y = 0$ is $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$.
If $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$, then $y(0) = c_1 + c_2$,

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Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$, and the characteristic roots are $\lambda = \dfrac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i$. It follows that the system is under-damped and that the general solution of $y'' + 6y' + 18y = 0$ is $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$.
If $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$, then $y(0) = c_1 + c_2$, so $y(0) = 0$ if and only if $c_2 = -c_1$.

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Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$, and the characteristic roots are $\lambda = \dfrac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i$. It follows that the system is under-damped and that the general solution of $y'' + 6y' + 18y = 0$ is $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$.
If $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$, then $y(0) = c_1 + c_2$, so $y(0) = 0$ if and only if $c_2 = -c_1$.
If $y(t) = c_1 \left(e^{(-3+3i)t} - e^{(-3-3i)t}\right)$, then $y'(t) = c_1 \left((-3+3i)e^{(-3+3i)t} - (-3-3i)e^{(-3-3i)t}\right)$ and $y'(0) = c_1\left((-3+3i) - (-3-3i)\right) = 6c_1 i$,

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Solution

The characteristic polynomial of $y'' + 6y' + 18y = 0$ is $\lambda^2 + 6\lambda + 18$, and the characteristic roots are $\lambda = \dfrac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i$. It follows that the system is under-damped and that the general solution of $y'' + 6y' + 18y = 0$ is $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$.
If $y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t}$, then $y(0) = c_1 + c_2$, so $y(0) = 0$ if and only if $c_2 = -c_1$.
If $y(t) = c_1 \left(e^{(-3+3i)t} - e^{(-3-3i)t}\right)$, then $y'(t) = c_1 \left((-3+3i)e^{(-3+3i)t} - (-3-3i)e^{(-3-3i)t}\right)$ and $y'(0) = c_1 \left((-3+3i) - (-3-3i)\right) = 6c_1 i$, so $y'(0) = 0.6$ if and only if $c_1 = \frac{1}{10i}$.

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Solution

So $y(t) = \frac{1}{10i}\left(e^{(-3+3i)t} - e^{(-3-3i)t}\right) = \frac{1}{5}e^{-3t}\sin(3t)$ is a particular solution $y(t)$ that satisfies the initial conditions $y(0) = 0$, $y'(0) = 0.6$.

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Solution

So $y(t) = \frac{1}{10i}\left(e^{(-3+3i)t} - e^{(-3-3i)t}\right) = \frac{1}{5}e^{-3t}\sin(3t)$ is a particular solution $y(t)$ that satisfies the initial conditions $y(0) = 0$, $y'(0) = 0.6$.
To find the steady-state solution of the equation $y'' + 6y' + 18y = 45\cos 3t$, we will first find the general solution of $y'' + 6y' + 18y = 45\cos 3t$.

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Solution

So $y(t) = \frac{1}{10i}\left(e^{(-3+3i)t} - e^{(-3-3i)t}\right) = \frac{1}{5}e^{-3t}\sin(3t)$ is a particular solution $y(t)$ that satisfies the initial conditions $y(0) = 0$, $y'(0) = 0.6$.

To find the steady-state solution of the equation $y'' + 6y' + 18y = 45\cos 3t$, we will first find the general solution of $y'' + 6y' + 18y = 45\cos 3t$. We have already seen that the general solution of $y'' + 6y' + 18y = 0$ is $y(t) = c_1e^{(-3+3i)t} + c_2e^{(-3-3i)t}$, so we just have to find a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.

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Solution

We will use the method of undetermined coefficient to find a particular solution of $y'' + 6y' + 18y = 45 \cos 3t$.

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Solution

We will use the method of undetermined coefficient to find a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.
Let $y(t) = A\cos 3t + B\sin 3t$.

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Solution

We will use the method of undetermined coefficient to find a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.
Let $y(t) = A\cos 3t + B\sin 3t$. Then

$$\begin{aligned} y''(t) + 6y'(t) + 18y(t) &= -9A\cos 3t - 9B\sin 3t \\ &\quad - 18A\sin 3t + 18B\cos 3t \\ &\quad + 18A\cos 3t + 18B\sin 3t \\ &= (9A + 18B)\cos 3t \\ &\quad + (-18A + 9B)\sin 3t, \end{aligned}$$

Solution

We will use the method of undetermined coefficient to find a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.
Let $y(t) = A\cos 3t + B\sin 3t$. Then

$$\begin{aligned} y''(t) + 6y'(t) + 18y(t) &= -9A\cos 3t - 9B\sin 3t \\ &\quad - 18A\sin 3t + 18B\cos 3t \\ &\quad + 18A\cos 3t + 18B\sin 3t \\ &= (9A + 18B)\cos 3t \\ &\quad + (-18A + 9B)\sin 3t, \end{aligned}$$

so $y(t) = A\cos 3t + B\sin 3t$ is a particular solution of $y'' + 6y' + 18y = 45\cos 3t$ if and only if $9A + 18B = 45$ and $-18A + 9B = 0$.

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Solution

The solution of the linear system

$$9A + 18B = 45$$
$$-18A + 9B = 0$$

is $A = 1$ and $B = 2$,

Solution

The solution of the linear system

$$\begin{aligned} 9A + 18B &= 45 \\ -18A + 9B &= 0 \end{aligned}$$

is $A = 1$ and $B = 2$, so $y(t) = \cos 3t + 2\sin 3t$ is a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.

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Solution

The solution of the linear system

$$9A + 18B = 45$$
$$-18A + 9B = 0$$

is $A = 1$ and $B = 2$, so $y(t) = \cos 3t + 2\sin 3t$ is a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.
It follows that the general solution of
$y'' + 6y' + 18y = 45\cos 3t$ is

$$y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t} + \cos 3t + 2\sin 3t.$$

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Solution

The solution of the linear system

$$9A + 18B = 45$$
$$-18A + 9B = 0$$

is $A = 1$ and $B = 2$, so $y(t) = \cos 3t + 2\sin 3t$ is a particular solution of $y'' + 6y' + 18y = 45\cos 3t$.
It follows that the general solution of $y'' + 6y' + 18y = 45\cos 3t$ is

$$y(t) = c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t} + \cos 3t + 2\sin 3t.$$

Since $c_1 e^{(-3+3i)t} + c_2 e^{(-3-3i)t} \to 0$ as $t \to \infty$, it follows that $y(t) = \cos 3t + 2\sin 3t$ is the steady-state solution of the equation $y'' + 6y' + 18y = 45\cos 3t$.

Plan for next week

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Plan for next week

Wednesday we shall

- study how to solve *systems of linear equations*,
- introduce *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

Section 1.1-1.2 in “Linear Algebras and Its Applications” (pages 1-23).

Plan for next week

Wednesday we shall

- study how to solve *systems of linear equations*,
- introduce *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

Section 1.1-1.2 in "Linear Algebras and Its Applications" (pages 1-23).

Thursday we shall introduce and study

- *vectors*,
- *linear combinations* of vectors,
- subsets *spanned* by vectors,
- *vector equations*.

Section 1.3 in "Linear Algebras and Its Applications" (pages 24-34).

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