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TMA4115 - Calculus 3

Lecture 5, Jan 30

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of the previous lecture

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Review of the previous lecture

Last time we

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Review of the previous lecture

Last time we

- studied second-order linear differential equations,

Review of the previous lecture

Last time we

- studied second-order linear differential equations,
- introduced the *Wronskian*,

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Review of the previous lecture

Last time we

- studied second-order linear differential equations,
- introduced the *Wronskian*,
- completely solved second-order homogeneous linear differential equations with constant coefficients.

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Today's lecture

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Today we shall

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- study harmonic motions,

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- study harmonic motions,
- study solutions of second-order linear inhomogeneous differential equations,

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Today's lecture

Today we shall

- study harmonic motions,
- study solutions of second-order linear inhomogeneous differential equations,
- look at the method of undetermined coefficients.

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Second-order homogeneous linear differential equations

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Second-order homogeneous linear differential equations

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

on the interval (α, β).

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Second-order homogeneous linear differential equations

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

on the interval (α, β). Then

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is the general solution of (1).

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Fundamental set of solutions

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.

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Fundamental set of solutions

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- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.

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- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either

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- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,

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Fundamental set of solutions

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- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,
  2. or by showing that the Wronskian of y_1 and y_2 is not zero at any point.

Homogeneous equations with constant coefficients

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Homogeneous equations with constant coefficients

Consider the second-order homogeneous linear differential equation

$$y'' + py' + qy = 0$$

with constant coefficients.

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Homogeneous equations with constant coefficients

Consider the second-order homogeneous linear differential equation

$$y'' + py' + qy = 0$$

with constant coefficients.

- The *characteristic polynomial* of the equation is the polynomial $\lambda^2 + p\lambda + q$.

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Homogeneous equations with constant coefficients

Consider the second-order homogeneous linear differential equation

$$y'' + py' + qy = 0$$

with constant coefficients.

- The *characteristic polynomial* of the equation is the polynomial $\lambda^2 + p\lambda + q$.
- The roots

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

of $\lambda^2 + p\lambda + q$ are called the *characteristic roots* of the equation.

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Homogeneous equations with constant coefficients

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Homogeneous equations with constant coefficients

- If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2, and the general solution of $y'' + py' + qy = 0$ is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

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Homogeneous equations with constant coefficients

- If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2, and the general solution of $y'' + py' + qy = 0$ is
$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$
- If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$, and the general solution of $y'' + py' + qy = 0$ is
$$y(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt).$$

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Homogeneous equations with constant coefficients

- If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root λ, and the general solution of $y'' + py' + qy = 0$ is

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

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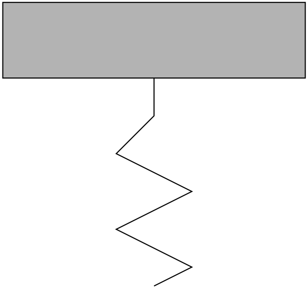
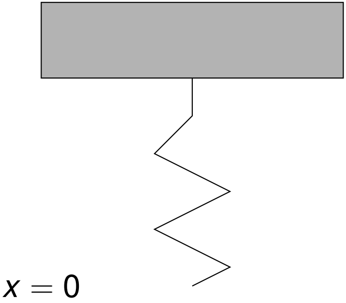
Harmonic motion



We consider a spring suspended from a beam.

Harmonic motion



We consider a spring suspended from a beam. The position of the bottom of the spring is the reference point from which we measure displacement, so it corresponds to $x = 0$.

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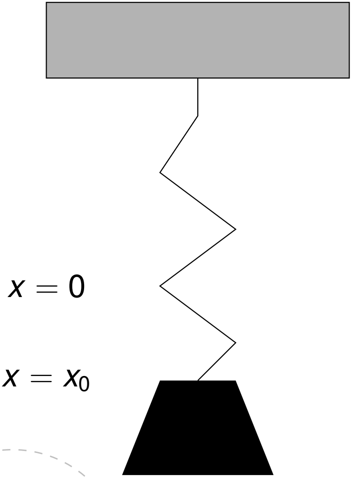
Harmonic motion



We consider a spring suspended from a beam. The position of the bottom of the spring is the reference point from which we measure displacement, so it corresponds to $x = 0$.
We then attach a weight of mass m to the spring. This weight stretches the spring until it is once more in equilibrium at $x = x_0$.

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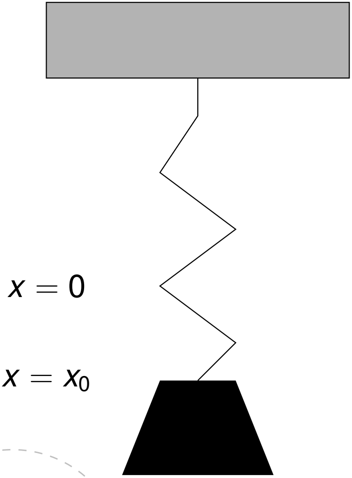
Harmonic motion



At this point there are two forces acting on the mass. There is the force of gravity mg, and there is the restoring force of the spring which we denote by $R(x)$ since it depends on the distance x that the spring is stretched.

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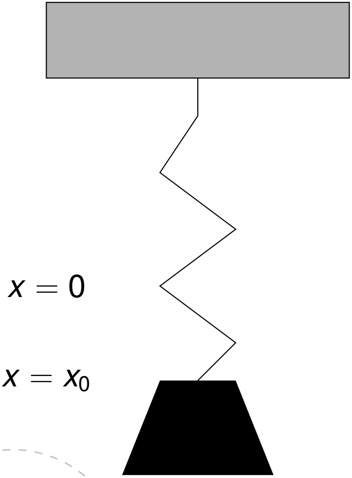
Harmonic motion



At this point there are two forces acting on the mass. There is the force of gravity mg, and there is the restoring force of the spring which we denote by $R(x)$ since it depends on the distance x that the spring is stretched.

Since we have equilibrium at $x = x_0$, the total force on the weight is 0, so $R(x_0) + mg = 0$.

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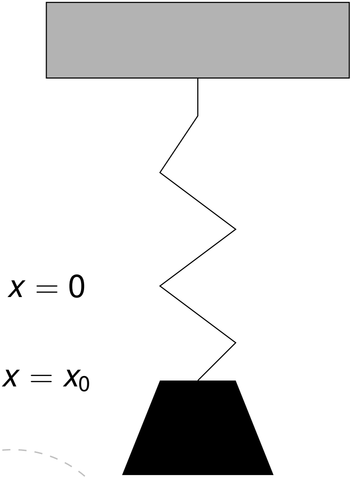
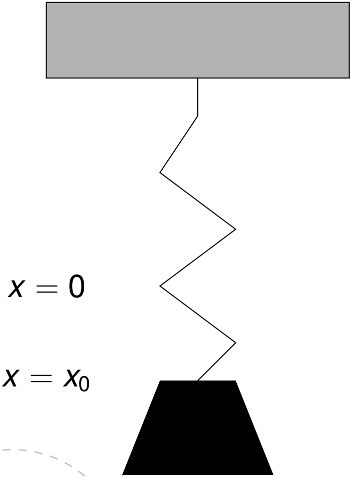
Harmonic motion



We now set the mass in motion by stretching the spring further.

Harmonic motion



We now set the mass in motion by stretching the spring further. In addition to gravity and the restoring force, there is a damping force *D* which is the resistance to the motion of the weight due to the medium through which the weight is moving and perhaps to something internal to the spring.

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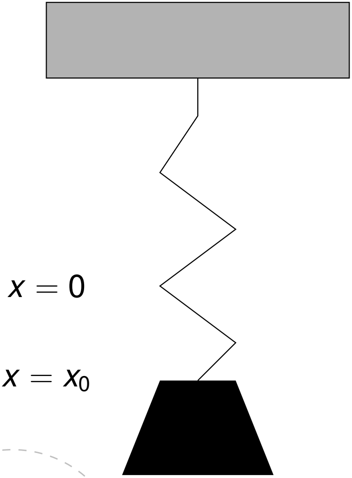
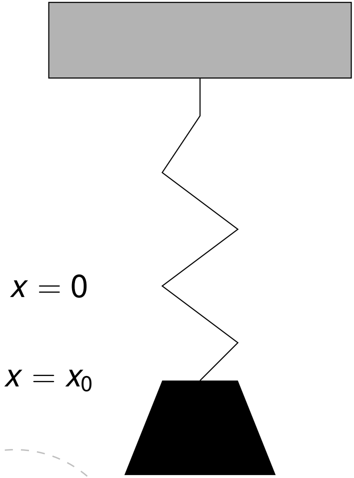
Harmonic motion



We assume that D depends on the velocity x' of the mass, and write it as $D(x')$.

Harmonic motion



We assume that D depends on the velocity x' of the mass, and write it as $D(x')$. According to Newton's second law, we have

$$mx'' = R(x) + mg + D(x').$$

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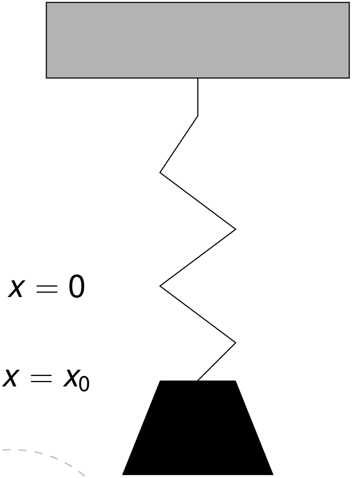
Harmonic motion



We assume that $R(x) = -kx$ for some positive constant k called the *spring constant*, and that $D(x') = -\mu x'$ for some nonnegative constant μ called the *damping constant*.

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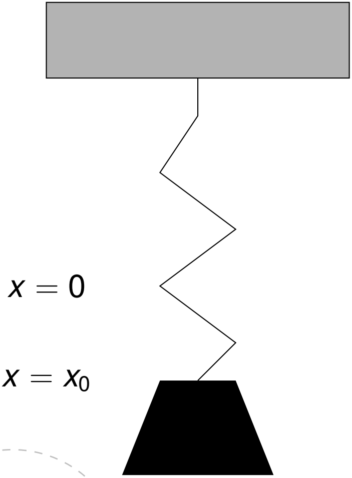
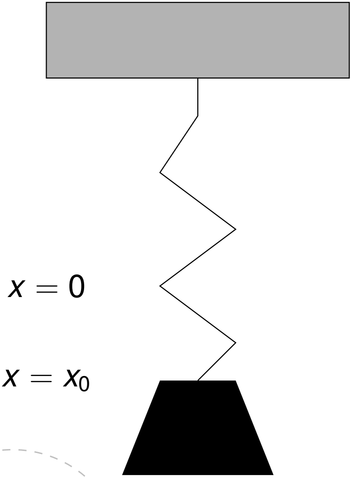
Harmonic motion



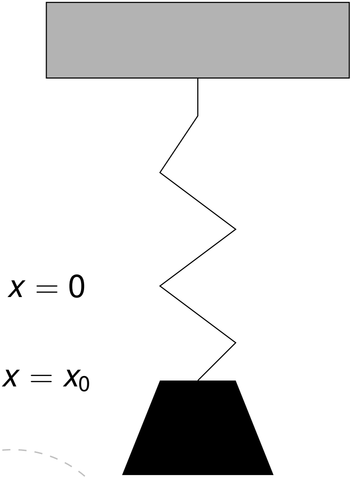
We assume that $R(x) = -kx$ for some positive constant k called the *spring constant*, and that $D(x') = -\mu x'$ for some nonnegative constant μ called the *damping constant*.
Thus we have
$mx'' = -kx + mg - \mu x'$.

Harmonic motion



Recall that $R(x_0) + mg = 0$.

Harmonic motion



Recall that $R(x_0) + mg = 0$. So $mg = -R(x_0) = kx_0$.

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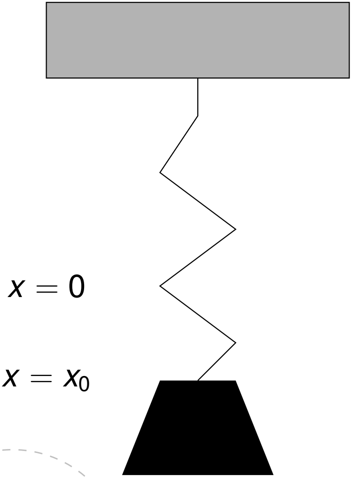
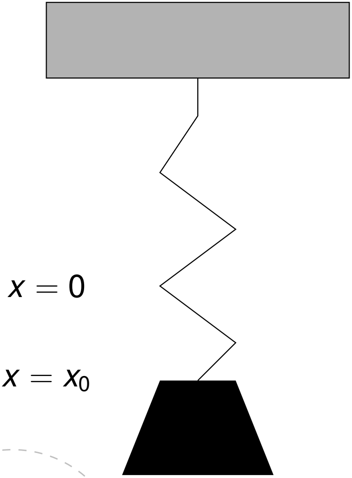


Recall that $R(x_0) + mg = 0$. So $mg = -R(x_0) = kx_0$. If we let $y = x - x_0$, then $mx'' = -kx + mg - \mu x'$ becomes $my'' + \mu y' + ky = 0$ because $x' = y'$ and $x'' = y''$.

Harmonic motion



If we let $\omega_0 = \sqrt{k/m}$ and $c = \mu/2m$, then the above equation becomes

$$y'' + 2cy' + \omega_0^2 y = 0$$

where $c \geq 0$ and $\omega_0 > 0$.

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Harmonic motion

The motion described by a solution to the equation

$$y'' + 2cy' + \omega_0^2 y = 0$$

where $c \geq 0$ and $\omega_0 > 0$, is called a *harmonic motion*.

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Simple harmonic motion

If $c = 0$ we say that the system is *undamped*.

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If $c = 0$ we say that the system is *undamped*. In that case, the equation becomes

$$y'' + \omega_0^2 y = 0$$

where $\omega_0 > 0$.
The general solution to this equation is

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

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The motion described by this solution is called a *simple harmonic motion*.

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The motion described by this solution is called a *simple harmonic motion*. The number ω_0 is called the *natural frequency*.

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The motion described by this solution is called a *simple harmonic motion*. The number ω_0 is called the *natural frequency*. The number $T = 2\pi/\omega_0$ is called the *period*.

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Amplitude and phase angle

It is frequently convenient to put the solution $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ into another form that is more convenient and more revealing of the nature of the solution.

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It is frequently convenient to put the solution $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ into another form that is more convenient and more revealing of the nature of the solution. Let $z = c_1 + ic_2$. If we let $A = |z|$ and $\phi = \mathrm{Arg}(z)$, then $z = A(\cos(\phi) + i\sin(\phi))$ from which it follows that $c_1 = A\cos(\phi)$ and $c_2 = A\sin(\phi)$,

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$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$
$$= A\cos(\phi)\cos(\omega_0 t) + A\sin(\phi)\sin(\omega_0 t) = A\cos(\omega_0 t - \phi).$$

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Amplitude and phase angle

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$$\begin{aligned} y(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= A\cos(\phi)\cos(\omega_0 t) + A\sin(\phi)\sin(\omega_0 t) = A\cos(\omega_0 t - \phi). \end{aligned}$$

The number A is called the *amplitude*, and the number ϕ is called the *phase*.

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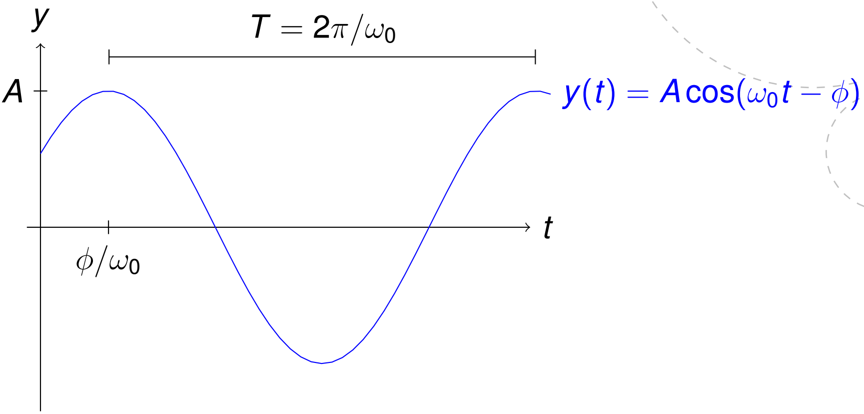
Simple harmonic motion



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The underdamped case

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The underdamped case

If $0 < c < \omega_0$, then the characteristic roots of $y'' + 2cy' + \omega_0^2 y = 0$ are

$$\lambda = \frac{-2c \pm \sqrt{(2c)^2 - 4\omega_0}}{2} = -c \pm \sqrt{c^2 - \omega_0^2} = -c \pm i\sqrt{\omega_0^2 - c^2}$$

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so the the general solution to $y'' + 2cy' + \omega_0^2 y = 0$ is

$$y(t) = e^{-ct}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$

where $\omega = \sqrt{\omega_0^2 - c^2}$.

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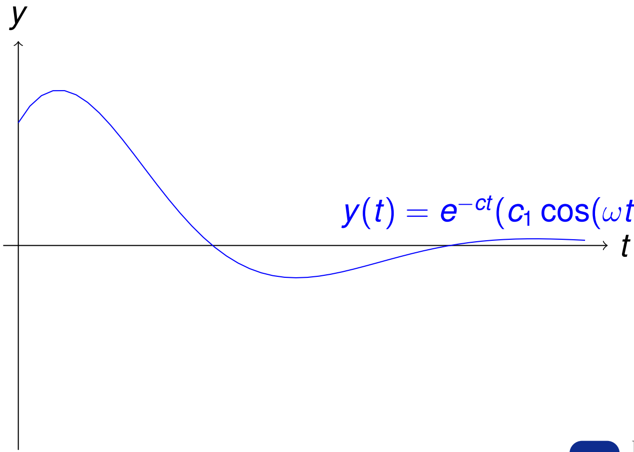
The underdamped case



$$y(t) = e^{-ct}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$

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The overdamped case

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The overdamped case

If $c > \omega_0$, then the characteristic roots of $y'' + 2cy' + \omega_0^2 y = 0$ are

$$\lambda = \frac{-2c \pm \sqrt{(2c)^2 - 4\omega_0}}{2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

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The overdamped case

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$$\lambda = \frac{-2c \pm \sqrt{(2c)^2 - 4\omega_0}}{2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

so the the general solution to $y'' + 2cy' + \omega_0^2 y = 0$ is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where $\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$ and $\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}$.

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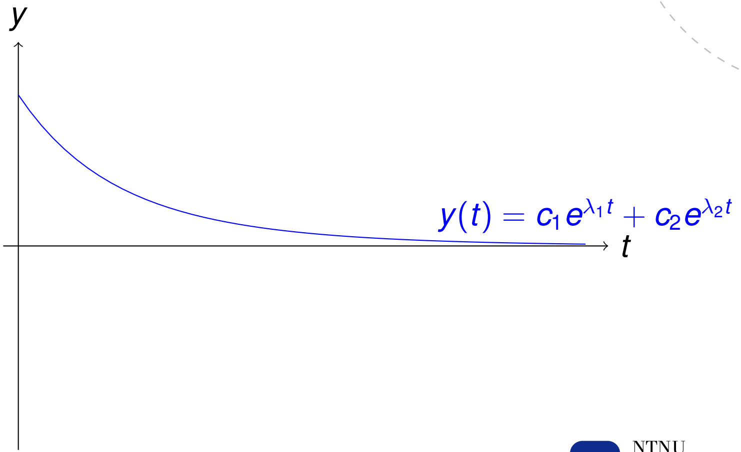
The overdamped case



The critically damped case

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The critically damped case

If $c = \omega_0$, then the equation $y'' + 2cy' + \omega_0^2 y = 0$ only has one characteristic root

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so the the general solution to $y'' + 2cy' + \omega_0^2 y = 0$ is

$$y(t) = c_1 e^{-ct} + c_2 t e^{-ct}.$$

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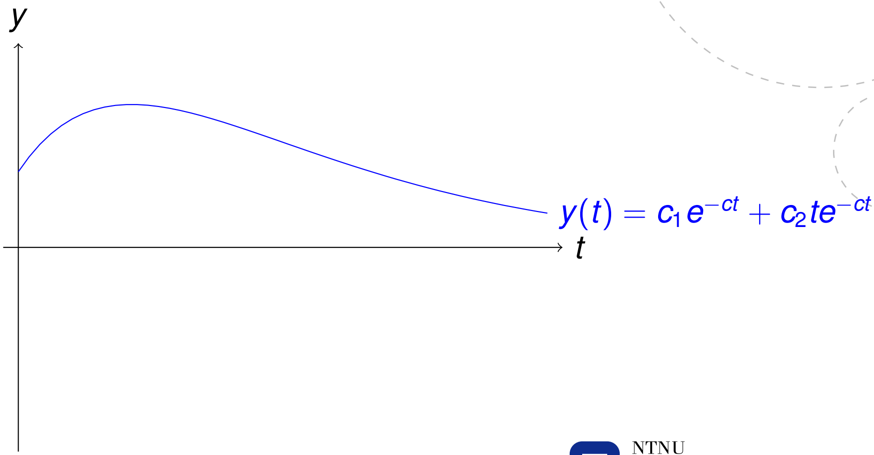
The critically damped case



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Inhomogeneous equations

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Inhomogeneous equations

We now turn to the solution of inhomogeneous second-order linear differential equations

$$y'' + py' + qy = f$$

where $p = p(t)$, $q = q(t)$ and $f = f(t)$ are functions of the independent variable.

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Inhomogeneous equations

Suppose we have found a particular solution y_p to the equation $y'' + py' + qy = f$.

Inhomogeneous equations

Suppose we have found a particular solution y_p to the equation $y'' + py' + qy = f$.
If y_h is a solution to the homogeneous equation $y'' + py' + qy = 0$,

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Inhomogeneous equations

Suppose we have found a particular solution y_p to the equation $y'' + py' + qy = f$.
If y_h is a solution to the homogeneous equation $y'' + py' + qy = 0$, then $y_p + y_h$ is a solution to the inhomogeneous equation $y'' + py' + qy = f$

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Inhomogeneous equations

Suppose we have found a particular solution y_p to the equation $y'' + py' + qy = f$.
If y_h is a solution to the homogeneous equation $y'' + py' + qy = 0$, then $y_p + y_h$ is a solution to the inhomogeneous equation $y'' + py' + qy = f$ because
$(y_p + y_h)'' + p(y_p + y_h)' + q(y_p + y_h) =$
$(y_p'' + py_p' + qy_p) + (y_h'' + py_h' + qy_h) = f + 0 = f$.

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Inhomogeneous equations

Conversely, if y_{p_1} and y_{p_2} are two different solutions to the inhomogeneous equation $y'' + py' + qy = f$, then $y_h = y_{p_1} - y_{p_2}$ is a solution to the homogeneous equation $y'' + py' + qy = 0$

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Inhomogeneous equations

Conversely, if y_{p_1} and y_{p_2} are two different solutions to the inhomogeneous equation $y'' + py' + qy = f$, then $y_h = y_{p_1} - y_{p_2}$ is a solution to the homogeneous equation $y'' + py' + qy = 0$ because
$(y_{p_1} - y_{p_2})'' + p(y_{p_1} - y_{p_2})' + q(y_{p_1} - y_{p_2}) =$
$(y''_{p_1} + py'_{p_1} + qy_{p_1}) - (y''_{p_2} + py'_{p_2} + qy_{p_2}) = f - f = 0.$

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Inhomogeneous equations

It follows that if y_p is a particular solution to the inhomogeneous equation $y'' + py' + qy = f$ and y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$, then the general solution to the inhomogeneous equation $y'' + py' + qy = f$ is

$$y = y_p + c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are constants.

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The method of undetermined coefficients

The method of undetermined coefficients

Consider the inhomogeneous second-order linear differential equation

$$y'' + py' + qy = f.$$

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The method of undetermined coefficients

Consider the inhomogeneous second-order linear differential equation

$$y'' + py' + qy = f.$$

If the function f has a form that is replicated under differentiation, then look for a solution with the same general form as f.

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Exponential forcing terms

Exponential forcing terms

If $f(t) = e^{at}$,

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Exponential forcing terms

If $f(t) = e^{at}$, then $f'(t) = ae^{at}$,

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Exponential forcing terms

If $f(t) = e^{at}$, then $f'(t) = ae^{at}$, so we will look for a solution of the form $y(t) = be^{at}$.

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Example

Let us find the general solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

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Example

Let us find the general solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

Let us first find a particular solution to the equation $y'' - y' - 2y = 2e^{-2t}$.

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Example

Let us find the general solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

Let us first find a particular solution to the equation
$y'' - y' - 2y = 2e^{-2t}$.
Since the right hand side is $2e^{-2t}$, we let $y(t) = ae^{-2t}$.

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Example

Let us find the general solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

Let us first find a particular solution to the equation $y'' - y' - 2y = 2e^{-2t}$.
Since the right hand side is $2e^{-2t}$, we let $y(t) = ae^{-2t}$.
Then $y'' - y' - 2y = 4ae^{-2t} - (-2)ae^{-2t} - 2ae^{-2t} = 4ae^{-2t}$.

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Example

Let us find the general solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

Let us first find a particular solution to the equation
$y'' - y' - 2y = 2e^{-2t}$.
Since the right hand side is $2e^{-2t}$, we let $y(t) = ae^{-2t}$.
Then $y'' - y' - 2y = 4ae^{-2t} - (-2)ae^{-2t} - 2ae^{-2t} = 4ae^{-2t}$.
So $y(t) = ae^{-2t}$ is a solution if and only if $4a = 2$.

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Example

Let us find the general solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

Let us first find a particular solution to the equation
$y'' - y' - 2y = 2e^{-2t}$.
Since the right hand side is $2e^{-2t}$, we let $y(t) = ae^{-2t}$.
Then $y'' - y' - 2y = 4ae^{-2t} - (-2)ae^{-2t} - 2ae^{-2t} = 4ae^{-2t}$.
So $y(t) = ae^{-2t}$ is a solution if and only if $4a = 2$.
Thus $y(t) = \frac{1}{2}e^{-2t}$ is a particular solution to the equation
$y'' - y' - 2y = 2e^{-2t}$.

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Example

We will then find the general solution of the homogeneous equation $y'' - y' - 2y = 0$.

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Example

We will then find the general solution of the homogeneous equation $y'' - y' - 2y = 0$. The characteristic polynomial is $\lambda^2 - \lambda - 2$,

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Example

We will then find the general solution of the homogeneous equation $y'' - y' - 2y = 0$. The characteristic polynomial is $\lambda^2 - \lambda - 2$, and the characteristic roots are

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}.$$

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Example

We will then find the general solution of the homogeneous equation $y'' - y' - 2y = 0$. The characteristic polynomial is $\lambda^2 - \lambda - 2$, and the characteristic roots are

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}.$$

So the general solution of $y'' - y' - 2y = 0$ is $y(t) = c_1 e^{2t} + c_2 e^{-t}$.

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Example

We will then find the general solution of the homogeneous equation $y'' - y' - 2y = 0$. The characteristic polynomial is $\lambda^2 - \lambda - 2$, and the characteristic roots are

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}.$$

So the general solution of $y'' - y' - 2y = 0$ is $y(t) = c_1 e^{2t} + c_2 e^{-t}$.

It follows that the general solution of $y'' - y' - 2y = 2e^{-2t}$ is $y(t) = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{2} e^{-2t}$.

Trigonometric forcing terms

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Trigonometric forcing terms

If

$$f(t) = A\cos(\omega t) + B\sin(\omega t),$$

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Trigonometric forcing terms

If

$$f(t) = A\cos(\omega t) + B\sin(\omega t),$$

then

$$f'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t),$$

Trigonometric forcing terms

If

$$f(t) = A\cos(\omega t) + B\sin(\omega t),$$

then

$$f'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t),$$

so we will look for a solution of the form

$$y(t) = a\cos(\omega t) + b\sin(\omega t).$$

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Let $y(t) = a\cos(3t) + b\sin(3t)$.

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Let $y(t) = a\cos(3t) + b\sin(3t)$. Then

$$\begin{aligned} y''(t) + 2y'(t) - 3y(t) &= -9a\cos(3t) - 9b\sin(3t) \\ &\quad + 2\bigl(-3a\sin(3t) + 3b\cos(3t)\bigr) \\ &\quad - 3a\cos(3t) - 3b\sin(3t) \\ &= (-9a + 6b - 3a)\cos(3t) \\ &\quad + (-9b - 6a - 3b)\sin(3t), \end{aligned}$$

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Let $y(t) = a\cos(3t) + b\sin(3t)$. Then

$$\begin{aligned} y''(t) + 2y'(t) - 3y(t) &= -9a\cos(3t) - 9b\sin(3t) \\ &\quad + 2\big(-3a\sin(3t) + 3b\cos(3t)\big) \\ &\quad - 3a\cos(3t) - 3b\sin(3t) \\ &= (-9a + 6b - 3a)\cos(3t) \\ &\quad + (-9b - 6a - 3b)\sin(3t), \end{aligned}$$

so $y(t) = a\cos(3t) + b\sin(3t)$ is a solution if and only if $-12a + 6b = 0$ and $-12b - 6 = 5$.

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Example

The solution to the system

$$-12a + 6b = 0$$
$$-6a - 12b = 5$$

is $a = \frac{5}{-30} = \frac{-1}{6}$ and $b = \frac{12a}{6} = 2a = \frac{-2}{6} = \frac{-1}{3}$,

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Example

The solution to the system

$$-12a + 6b = 0$$
$$-6a - 12b = 5$$

is $a = \frac{5}{-30} = \frac{-1}{6}$ and $b = \frac{12a}{6} = 2a = \frac{-2}{6} = \frac{-1}{3}$, so $y(t) = -\frac{1}{6}\cos(3t) - \frac{1}{3}\sin(3t)$ is a particular solution to the equation $y'' + 2y' - 3y = 5\sin(3t)$.

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The complex method

The complex method

There is another way to find a particular solution in situations where

$$f(t) = A\cos(\omega t) + B\sin(\omega t).$$

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The complex method

There is another way to find a particular solution in situations where

$$f(t) = A\cos(\omega t) + B\sin(\omega t).$$

If $z(t)$ is a solution of the equation

$$y'' + py' + qy = e^{\omega it},$$

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The complex method

There is another way to find a particular solution in situations where

$$f(t) = A\cos(\omega t) + B\sin(\omega t).$$

If $z(t)$ is a solution of the equation

$$y'' + py' + qy = e^{\omega it},$$

then a suitable linear combination of $\mathrm{Re}(z(t))$ and $\mathrm{Im}(z(t))$ will be a solution to

$$y'' + py' + qy = A\cos(\omega t) + B\sin(\omega t).$$

Example

Consider again the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

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Example

Consider again the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Since $5\sin(3t) = \text{Im}(5e^{3it})$, it follows that if $z(t)$ is a solution to $z'' + 2z' - 3z = 5e^{3it}$, then $y(t) = \text{Im}(z(t))$ is a solution to $y'' + 2y' - 3y = 5\sin(3t)$.

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Example

Consider again the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Since $5\sin(3t) = \mathrm{Im}(5e^{3it})$, it follows that if $z(t)$ is a solution to $z'' + 2z' - 3z = 5e^{3it}$, then $y(t) = \mathrm{Im}(z(t))$ is a solution to $y'' + 2y' - 3y = 5\sin(3t)$.
To find a solution to $z'' + 2z' - 3z = 5e^{3it}$, we let $z(t) = ae^{3it}$.

Example

Consider again the equation

$$y'' + 2y' - 3y = 5\sin(3t).$$

Since $5\sin(3t) = \text{Im}(5e^{3it})$, it follows that if $z(t)$ is a solution to $z'' + 2z' - 3z = 5e^{3it}$, then $y(t) = \text{Im}(z(t))$ is a solution to $y'' + 2y' - 3y = 5\sin(3t)$.
To find a solution to $z'' + 2z' - 3z = 5e^{3it}$, we let $z(t) = ae^{3it}$.
Then

$$\begin{aligned} z''(t) + 2z'(t) - 3z(t) &= -9ae^{3it} + 6iae^{3it} - 3ae^{3it} \\ &= (-12 + 6i)ae^{3it}, \end{aligned}$$

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Example

so $z(t) = ae^{3it}$ is a solution of $z'' + 2z' - 3z = 5e^{3it}$ if and only if $a = \frac{5}{-12+6i} = \frac{-1}{6}\frac{5}{2-i} = \frac{-5(2+i)}{6(2-i)(2+i)} = \frac{-10-5i}{30} = \frac{-2-i}{6}$.

Example

so $z(t) = ae^{3it}$ is a solution of $z'' + 2z' - 3z = 5e^{3it}$ if and only if $a = \frac{5}{-12+6i} = \frac{-1}{6}\frac{5}{2-i} = \frac{-5(2+i)}{6(2-i)(2+i)} = \frac{-10-5i}{30} = \frac{-2-i}{6}$.
So

$$\begin{aligned} z(t) &= \tfrac{-2-i}{6} e^{3it} = \tfrac{-2-i}{6}\big(\cos(3t) + i\sin(3t)\big) \\ &= \tfrac{-1}{3}\cos(3t) + \tfrac{1}{6}\sin(3t) + i\big(\tfrac{-1}{6}\cos(3t) + \tfrac{-1}{3}\sin(3t)\big) \end{aligned}$$

is a solution of $z'' + 2z' - 3z = 5e^{3it}$,

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Example

so $z(t) = ae^{3it}$ is a solution of $z'' + 2z' - 3z = 5e^{3it}$ if and only if $a = \frac{5}{-12+6i} = \frac{-1}{6}\frac{5}{2-i} = \frac{-5(2+i)}{6(2-i)(2+i)} = \frac{-10-5i}{30} = \frac{-2-i}{6}$.
So

$$\begin{aligned} z(t) &= \tfrac{-2-i}{6} e^{3it} = \tfrac{-2-i}{6}\big(\cos(3t) + i\sin(3t)\big) \\ &= \tfrac{-1}{3}\cos(3t) + \tfrac{1}{6}\sin(3t) + i\big(\tfrac{-1}{6}\cos(3t) + \tfrac{-1}{3}\sin(3t)\big) \end{aligned}$$

is a solution of $z'' + 2z' - 3z = 5e^{3it}$, and
$y(t) = \operatorname{Im}(z(t)) = \frac{-1}{6}\cos(3t) + \frac{-1}{3}\sin(3t)$ is a solution to
$y'' + 2y' - 3y = 5\sin(3t)$.

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Polynomial forcing terms

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Polynomial forcing terms

If

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots a_1 t + a_0,$$

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Polynomial forcing terms

If

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots a_1 t + a_0,$$

then

$$f'(t) = n a_n t^{n-1} + (n-1) a_{n-1} t^{n-2} + \ldots a_1,$$

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Polynomial forcing terms

If

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots a_1 t + a_0,$$

then

$$f'(t) = n a_n t^{n-1} + (n-1) a_{n-1} t^{n-2} + \dots a_1,$$

so we will look for a solution of the form

$$y(t) = b_n t^n + b_{n-1} t^{n-1} + \dots b_1 t + b_0.$$

Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 3t + 4.$$

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 3t + 4.$$

Let $y(t) = at + b$.

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 3t + 4.$$

Let $y(t) = at + b$. Then

$$y''(t) + 2y'(t) - 3y(t) = 2a - 3at - 3b = -3at + 2a - 3b$$

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 3t + 4.$$

Let $y(t) = at + b$. Then

$$y''(t) + 2y'(t) - 3y(t) = 2a - 3at - 3b = -3at + 2a - 3b$$

so $y(t) = at + b$ is a solution if and only if $a = -1$ and $b = \frac{4-2a}{-3} = -2$.

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Example

Let us find a particular solution to the equation

$$y'' + 2y' - 3y = 3t + 4.$$

Let $y(t) = at + b$. Then

$$y''(t) + 2y'(t) - 3y(t) = 2a - 3at - 3b = -3at + 2a - 3b$$

so $y(t) = at + b$ is a solution if and only if $a = -1$ and $b = \frac{4-2a}{-3} = -2$. Thus $y(t) = -t - 2$ is a particular solution to the equation $y'' + 2y' - 3y = 3t + 4$.

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Exceptional cases

Exceptional cases

The method of undetermined coefficients looks straightforward.

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Exceptional cases

The method of undetermined coefficients looks straightforward. There are, however, some exceptional cases to look out for.

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Exceptional cases

The method of undetermined coefficients looks straightforward. There are, however, some exceptional cases to look out for. If the forcing term f, and hence the proposed solution, is a solution to the homogeneous equation $y'' + py' + qy = 0$, then the proposed solution wouldn't work.

Exceptional cases

The method of undetermined coefficients looks straightforward. There are, however, some exceptional cases to look out for. If the forcing term f, and hence the proposed solution, is a solution to the homogeneous equation $y'' + py' + qy = 0$, then the proposed solution wouldn't work. Instead we have to multiply the proposed solution by t.

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = 3e^{-t}.$$

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = 3e^{-t}.$$

Let $y(t) = ae^{-t}$.

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = 3e^{-t}.$$

Let $y(t) = ae^{-t}$. Then

$$y''(t) - y'(t) - 2y(t) = ae^{-t} + ae^{-t} - 2ae^{-t} = 0,$$

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = 3e^{-t}.$$

Let $y(t) = ae^{-t}$. Then

$$y''(t) - y'(t) - 2y(t) = ae^{-t} + ae^{-t} - 2ae^{-t} = 0,$$

so it is not possible to find an a such that $y(t) = ae^{-t}$ becomes a solution to $y'' - y' - 2y = 3e^{-t}$.

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Example

Instead, we let $y(t) = ate^{-t}$.

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Example

Instead, we let $y(t) = ate^{-t}$. Then
$y'(t) = ae^{-t} - ate^{-t} = a(1-t)e^{-t}$,

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Example

Instead, we let $y(t) = ate^{-t}$. Then
$y'(t) = ae^{-t} - ate^{-t} = a(1 - t)e^{-t}$,
$y''(t) = -ae^{-t} - a(1 - t)e^{-t} = a(t - 2)e^{-t}$,

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Example

Instead, we let $y(t) = ate^{-t}$. Then
$y'(t) = ae^{-t} - ate^{-t} = a(1-t)e^{-t}$,
$y''(t) = -ae^{-t} - a(1-t)e^{-t} = a(t-2)e^{-t}$, and

$$\begin{aligned} y''(t) - y'(t) - 2y(t) &= a(t-2)e^{-t} - a(1-t)e^{-t} - 2ate^{-t} \\ &= -3ae^{-t}. \end{aligned}$$

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Example

Instead, we let $y(t) = ate^{-t}$. Then
$y'(t) = ae^{-t} - ate^{-t} = a(1-t)e^{-t}$,
$y''(t) = -ae^{-t} - a(1-t)e^{-t} = a(t-2)e^{-t}$, and

$$\begin{aligned} y''(t) - y'(t) - 2y(t) &= a(t-2)e^{-t} - a(1-t)e^{-t} - 2ate^{-t} \\ &= -3ae^{-t}. \end{aligned}$$

So $y(t) = -te^{-t}$ is a particular solution to the equation
$y'' - y' - 2y = 3e^{-t}$.

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Combination forcing terms

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Combination forcing terms

If y_f is a solution the differential equation $y'' + py' + qy = f$,

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Combination forcing terms

If y_f is a solution the differential equation $y'' + py' + qy = f$,
y_g is a solution the differential equation $y'' + py' + qy = g$,

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Combination forcing terms

If y_f is a solution the differential equation $y'' + py' + qy = f$,
y_g is a solution the differential equation $y'' + py' + qy = g$,
and c_1 and c_2 are constants,

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Combination forcing terms

If y_f is a solution the differential equation $y'' + py' + qy = f$, y_g is a solution the differential equation $y'' + py' + qy = g$, and c_1 and c_2 are constants, then

$$y(t) = c_1 y_f(t) + c_2 y_g(t)$$

is a solution to the differential equation

$$y'' + py' + qy = c_1 f + c_2 g.$$

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = e^{-2t} - 3e^{-t}.$$

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = e^{-2t} - 3e^{-t}.$$

We have already seen that $y_1(t) = \frac{1}{2}e^{-2t}$ is a solution to $y'' - y' - 2y = 2e^{-2t}$,

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = e^{-2t} - 3e^{-t}.$$

We have already seen that $y_1(t) = \frac{1}{2}e^{-2t}$ is a solution to $y'' - y' - 2y = 2e^{-2t}$, and that $y_2(t) = -te^{-t}$ is a solution to $y'' - y' - 2y = 3e^{-t}$.

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Example

Let us find a particular solution to the equation

$$y'' - y' - 2y = e^{-2t} - 3e^{-t}.$$

We have already seen that $y_1(t) = \frac{1}{2}e^{-2t}$ is a solution to $y'' - y' - 2y = 2e^{-2t}$, and that $y_2(t) = -te^{-t}$ is a solution to $y'' - y' - 2y = 3e^{-t}$.
It follows that $y(t) = \frac{1}{2}y_1(t) - y_2(t) = \frac{1}{4}e^{-2t} + te^{-t}$ is a particular solution to the equation $y'' - y' - 2y = e^{-2t} - 3e^{-t}$.

Plan for tomorrow

Tomorrow we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

Section 4.6 and 4.7 in “Second-Order Equations” (pages pages lxxii–lxxxvi).

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