NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 4, Jan 24

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of yesterday's lecture

NTNU
Norwegian University of
Science and Technology

Review of yesterday's lecture

Yesterday we

Review of yesterday's lecture

Yesterday we

- looked at how to use complex numbers to solve polynomial equations,

NTNU
Norwegian University of Science and Technology

Review of yesterday's lecture

Yesterday we

- looked at how to use complex numbers to solve polynomial equations,
- looked at *the fundamental theorem of algebra*,

NTNU
Norwegian University of Science and Technology

Review of yesterday’s lecture

Yesterday we

- looked at how to use complex numbers to solve polynomial equations,
- looked at *the fundamental theorem of algebra*,
- introduced *the complex exponential function*,

NTNU
Norwegian University of Science and Technology

Review of yesterday's lecture

Yesterday we

- looked at how to use complex numbers to solve polynomial equations,
- looked at *the fundamental theorem of algebra*,
- introduced *the complex exponential function*,
- and studied extensions of trigonometric functions to the complex numbers.

NTNU
Norwegian University of Science and Technology

Today's lecture

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall

Today's lecture

Today we shall

- study second-order linear differential equations,

Today's lecture

Today we shall

- study second-order linear differential equations,
- introduce the *Wronskian*,

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall

- study second-order linear differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.

NTNU
Norwegian University of Science and Technology

Second-order differential equations

NTNU
Norwegian University of
Science and Technology

Second-order differential equations

A second-order differential equation is a differential equation which can be written on the form

$$y'' = f(t, y, y').$$

NTNU
Norwegian University of Science and Technology

Second-order differential equations

A second-order differential equation is a differential equation which can be written on the form

$$y'' = f(t, y, y').$$

A solution to such an equation is a twice continuously differentiable function $y(t)$ satisfying

$$y''(t) = f(t, y(t), y'(t)).$$

NTNU
Norwegian University of Science and Technology

Examples of second-order differential equations

NTNU
Norwegian University of
Science and Technology

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$.

NTNU
Norwegian University of Science and Technology

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$.
- $y'' + 5y = 0$.

NTNU
Norwegian University of
Science and Technology

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$.
- $y'' + 5y = 0$.
- $t^2 y'' + \sin(t) y = 3$.

NTNU
Norwegian University of Science and Technology

Second-order linear differential equations

NTNU
Norwegian University of
Science and Technology

Second-order linear differential equations

A second-order linear differential equation is a differential equation with can be written on the form

$$y'' + p(t)y' + q(t)y = g(t).$$

NTNU
Norwegian University of Science and Technology

Examples of second-order differential equations

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$ is not linear.

NTNU
Norwegian University of Science and Technology

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$ is not linear.
- $y'' + 5y = 0$ is linear.

NTNU
Norwegian University of Science and Technology

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$ is not linear.
- $y'' + 5y = 0$ is linear.
- $t^3 y'' + \sin(t)y = 3$ is linear.

NTNU
Norwegian University of Science and Technology

Existence and uniqueness of solutions to a second-order linear equation

NTNU
Norwegian University of Science and Technology

Existence and uniqueness of solutions to a second-order linear equation

Suppose the functions p, q and g are continuous on the interval (α, β).

NTNU
Norwegian University of Science and Technology

Existence and uniqueness of solutions to a second-order linear equation

Suppose the functions p, q and g are continuous on the interval (α, β). Let t_0 be any point in (α, β).

NTNU
Norwegian University of Science and Technology

Existence and uniqueness of solutions to a second-order linear equation

Suppose the functions p, q and g are continuous on the interval (α, β). Let t_0 be any point in (α, β). Then for any real numbers a and b there is one and only one function defined on (α, β) which is a solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

on (α, β) and satisfies the initial conditions $y(t_0) = a$ and $y'(t_0) = b$.

Second-order linear homogeneous differential equations

NTNU
Norwegian University of Science and Technology

Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$.

NTNU
Norwegian University of
Science and Technology

Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$. If $g \neq 0$, then the differential equation is called *inhomogeneous* or *nonhomegeneous*.

Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$. If $g \neq 0$, then the differential equation is called *inhomogeneous* or *nonhomegeneous*.

Examples

NTNU
Norwegian University of Science and Technology

Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$. If $g \neq 0$, then the differential equation is called *inhomogeneous* or *nonhomegeneous*.

Examples

- $y'' + 5y = 0$ is homogeneous.

NTNU
Norwegian University of Science and Technology

Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$. If $g \neq 0$, then the differential equation is called *inhomogeneous* or *nonhomegeneous*.

Examples

- $y'' + 5y = 0$ is homogeneous.
- $t^3 y'' + \sin(t)y = 3$ is inhomogeneous.

NTNU
Norwegian University of Science and Technology

The superposition principle

NTNU
Norwegian University of
Science and Technology

The superposition principle

If y_1 and y_2 are solutions to the second-order homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then so is $y(t) = c_1 y_1(t) + c_2 y_2(t)$ for any choice of constants c_1 and c_2.

NTNU
Norwegian University of Science and Technology

Linear combinations

NTNU
Norwegian University of
Science and Technology

Linear combinations

- When y_1 and y_2 are functions and c_1 and c_2 are constants, then the function $c_1 y_1(t) + c_2 y_2(t)$ is a *linear combination* of y_1 and y_2.

NTNU
Norwegian University of Science and Technology

Linear combinations

- When y_1 and y_2 are functions and c_1 and c_2 are constants, then the function $c_1 y_1(t) + c_2 y_2(t)$ is a *linear combination* of y_1 and y_2.
- So the previous results says that if y_1 and y_2 are solutions to a second-order homogeneous linear differential equation, then any linear combination of y_1 and y_2 is also a solution to the same differential equation.

NTNU
Norwegian University of Science and Technology

Example

NTNU
Norwegian University of Science and Technology

Example

Let us find the solution of the differential equation

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0 \qquad (1)$$

on the interval $(0, \infty)$ which satisfies that $y(2) = 8$ and $y'(2) = 0$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the solution of the differential equation

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0 \tag{1}$$

on the interval $(0, \infty)$ which satisfies that $y(2) = 8$ and $y'(2) = 0$.

Let $y(t) = t^n$. Then

$$\begin{aligned} y''(t) - \frac{4}{t}y'(t) + \frac{6}{t^2}y(t) &= n(n-1)t^{n-2} - 4nt^{n-2} + 6t^{n-2} \\ &= (n^2 - 5n + 6)t^{n-2} \\ &= (n-2)(n-3)t^{n-2}. \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Example

Let us find the solution of the differential equation

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0 \tag{1}$$

on the interval $(0, \infty)$ which satisfies that $y(2) = 8$ and $y'(2) = 0$.

Let $y(t) = t^n$. Then

$$\begin{aligned} y''(t) - \frac{4}{t}y'(t) + \frac{6}{t^2}y(t) &= n(n-1)t^{n-2} - 4nt^{n-2} + 6t^{n-2} \\ &= (n^2 - 5n + 6)t^{n-2} \\ &= (n-2)(n-3)t^{n-2}. \end{aligned}$$

So y is a solution of (1) if and only if $n = 2$ or $n = 3$.

NTNU
Norwegian University of Science and Technology

Example

Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants.

NTNU
Norwegian University of
Science and Technology

Example

Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then it follows from the superposition principle and the calculations above that y is a solution of (1).

NTNU
Norwegian University of Science and Technology

Example

Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then it follows from the superposition principle and the calculations above that y is a solution of (1).

$$
\begin{aligned}
y'(t) &= 2c_1 t + 3c_2 t^2, \\
y(2) &= 4c_1 + 8c_2, \\
y'(2) &= 4c_1 + 12c_2,
\end{aligned}
$$

Example

Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then it follows from the superposition principle and the calculations above that y is a solution of (1).

$$\begin{aligned} y'(t) &= 2c_1 t + 3c_2 t^2, \\ y(2) &= 4c_1 + 8c_2, \\ y'(2) &= 4c_1 + 12c_2, \end{aligned}$$

so $y(2) = 8$ and $y'(2) = 0$ if and only if $4c_1 + 8c_2 = 8$ and $4c_1 + 12c_2 = 0$.

Example

The solution of the linear system

$$\begin{aligned} 4c_1 + 8c_2 &= 8 \\ 4c_1 + 12c_2 &= 0 \end{aligned}$$

is $c_1 = 6$ and $c_2 = -2$,

NTNU
Norwegian University of
Science and Technology

Example

The solution of the linear system

$$\begin{aligned} 4c_1 + 8c_2 &= 8 \\ 4c_1 + 12c_2 &= 0 \end{aligned}$$

is $c_1 = 6$ and $c_2 = -2$, so the solution of (1) which satisfies that $y(2) = 8$ and $y'(2) = 0$ is $y(t) = 6t^2 - 2t^3$.

NTNU
Norwegian University of Science and Technology

Example

Let $t_0 \in (0, \infty)$ and $a, b \in \mathbb{R}$.

NTNU
Norwegian University of
Science and Technology

Example

Let $t_0 \in (0, \infty)$ and $a, b \in \mathbb{R}$. Let us now find the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$.

NTNU
Norwegian University of Science and Technology

Example

Let $t_0 \in (0, \infty)$ and $a, b \in \mathbb{R}$. Let us now find the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$.
Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants.

NTNU
Norwegian University of
Science and Technology

Example

Let $t_0 \in (0, \infty)$ and $a, b \in \mathbb{R}$. Let us now find the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$.
Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then

$$y(t_0) = c_1 t_0^2 + c_2 t_0^3,$$
$$y'(t_0) = 2c_1 t_0 + 3c_2 t_0^2,$$

NTNU
Norwegian University of Science and Technology

Example

Let $t_0 \in (0, \infty)$ and $a, b \in \mathbb{R}$. Let us now find the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$.
Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then

$$y(t_0) = c_1 t_0^2 + c_2 t_0^3,$$
$$y'(t_0) = 2c_1 t_0 + 3c_2 t_0^2,$$

so $y(t_0) = a$ and $y'(t_0) = b$. if and only if $c_1 t_0^2 + c_2 t_0^3 = a$ and $2c_1 t_0 + 3c_2 t_0^2 = b$.

NTNU
Norwegian University of
Science and Technology

Example

The solution of the linear system

$$c_1 t_0^2 + c_2 t_0^3 = a$$
$$2c_1 t_0 + 3c_2 t_0^2 = b$$

is $c_1 = \frac{3at_0^2 - bt_0^3}{t_0^4}$ and $c_2 = \frac{bt_0^2 - 2at_0}{t_0^4}$,

NTNU
Norwegian University of Science and Technology

Example

The solution of the linear system

$$c_1 t_0^2 + c_2 t_0^3 = a$$
$$2c_1 t_0 + 3c_2 t_0^2 = b$$

is $c_1 = \frac{3at_0^2 - bt_0^3}{t_0^4}$ and $c_2 = \frac{bt_0^2 - 2at_0}{t_0^4}$, so the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$ is $y(t) = \frac{3at_0^2 - bt_0^3}{t_0^4} t^2 - \frac{bt_0^2 - 2at_0}{t_0^4} t^3$.

NTNU
Norwegian University of Science and Technology

Example

The solution of the linear system

$$c_1 t_0^2 + c_2 t_0^3 = a$$
$$2c_1 t_0 + 3c_2 t_0^2 = b$$

is $c_1 = \frac{3at_0^2 - bt_0^3}{t_0^4}$ and $c_2 = \frac{bt_0^2 - 2at_0}{t_0^4}$, so the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$ is
$y(t) = \frac{3at_0^2 - bt_0^3}{t_0^4} t^2 - \frac{bt_0^2 - 2at_0}{t_0^4} t^3$.
Notice that $t_0^4 = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$ where $y_1(t) = t^2$ and $y_2(t) = t^3$.

NTNU
Norwegian University of Science and Technology

The Wronskian

Let u and v be two differential functions. The *Wronskian* of u and v is the function

$$W(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t).$$

NTNU
Norwegian University of
Science and Technology

Solutions to initial value problems

NTNU
Norwegian University of
Science and Technology

Solutions to initial value problems

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

on the interval (α, β).

NTNU
Norwegian University of Science and Technology

Solutions to initial value problems

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

on the interval (α, β). Let t_0 be a point in the interval (α, β) and let a and b be arbitrary real numbers.

Solutions to initial value problems

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

on the interval (α, β). Let t_0 be a point in the interval (α, β) and let a and b be arbitrary real numbers. If $W(t)$ is the Wronskian of y_1 and y_2 and $W(t_0) \neq 0$,

NTNU
Norwegian University of
Science and Technology

Solutions to initial value problems

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

on the interval (α, β). Let t_0 be a point in the interval (α, β) and let a and b be arbitrary real numbers. If $W(t)$ is the Wronskian of y_1 and y_2 and $W(t_0) \neq 0$, then there exist constants c_1 and c_2 such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the unique solution to (2) on (α, β) which satisfies $y(t_0) = a$ and $y'(t_0) = b$.

Proof

NTNU
Norwegian University of
Science and Technology

Proof

It follows from the superposition principle that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) for any choice of constants c_1 and c_2.

NTNU
Norwegian University of Science and Technology

Proof

It follows from the superposition principle that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) for any choice of constants c_1 and c_2.

$y(t_0) = a$ and $y'(t_0) = b$ if and only if $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.

NTNU
Norwegian University of Science and Technology

Proof

It follows from the superposition principle that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) for any choice of constants c_1 and c_2.
$y(t_0) = a$ and $y'(t_0) = b$ if and only if $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.
Since $y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0) = W(t_0) \neq 0$,

NTNU
Norwegian University of Science and Technology

Proof

It follows from the superposition principle that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) for any choice of constants c_1 and c_2.

$y(t_0) = a$ and $y'(t_0) = b$ if and only if $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.

Since $y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = W(t_0) \neq 0$, the system

$$
\begin{aligned}
c_1 y_1(t_0) + c_2 y_2(t_0) &= a \\
c_1 y_1'(t_0) + c_2 y_2'(t_0) &= b
\end{aligned}
$$

has a solution,

NTNU
Norwegian University of Science and Technology

Proof

It follows from the superposition principle that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) for any choice of constants c_1 and c_2.

$y(t_0) = a$ and $y'(t_0) = b$ if and only if $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.

Since $y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = W(t_0) \neq 0$, the system

$$c_1 y_1(t_0) + c_2 y_2(t_0) = a$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$$

has a solution, so there exist constants c_1 and c_2 such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) on (α, β) which satisfies $y(t_0) = a$ and $y'(t_0) = b$.

NTNU
Norwegian University of Science and Technology

Proposition 1.26

NTNU
Norwegian University of
Science and Technology

Proposition 1.26

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β).

NTNU
Norwegian University of Science and Technology

Proposition 1.26

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then the Wronskian of y_1 and y_2 is either identically equal to zero on the interval (α, β), or it is never equal to zero on the interval (α, β).

Proof

NTNU
Norwegian University of
Science and Technology

Proof

If we differentiate the Wronskian W we get

$$\begin{aligned} W'(t) &= (y_1(t)y_2'(t) - y_1'(t)y_2(t))' \\ &= y_1'(t)y_2'(t) + y_1(t)y_2''(t) - y_1''(t)y_2(t) - y_1'(t)y_2'(t) \\ &= y_1(t)y_2''(t) - y_1''(t)y_2(t). \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Proof

If we differentiate the Wronskian W we get

$$\begin{aligned} W'(t) &= (y_1(t)y_2'(t) - y_1'(t)y_2(t))' \\ &= y_1'(t)y_2'(t) + y_1(t)y_2''(t) - y_1''(t)y_2(t) - y_1'(t)y_2'(t) \\ &= y_1(t)y_2''(t) - y_1''(t)y_2(t). \end{aligned}$$

Since y_1 and y_2 are solutions to the differential equation $y'' + py' + qy = 0$, we have that $y_1''(t) = -p(t)y_1'(t) - q(t)y_1$ and $y_2''(t) = -p(t)y_2'(t) - q(t)y_2$.

NTNU
Norwegian University of Science and Technology

Proof

It follows that

$$\begin{aligned} W'(t) &= y_1(t)y_2''(t) - y_1''(t)y_2(t) \\ &= -p(t)y_1(t)y_2'(t) - q(t)y_1(t)y_2(t) \\ &\quad + p(t)y_1'(t)y_2(t) + q(t)y_1(t)y_2(t) \\ &= -p(t)y_1(t)y_2'(t) + p(t)y_1'(t)y_2(t) \\ &= -p(t)({}_1(t)y_2'(t) - y_1'(t)y_2(t)) \\ &= -p(t)W(t), \end{aligned}$$

so W is a solution to the first-order differential equation

$$W' = -pW.$$

Proof

If t_0 is a point in (α, β), the solution to this equation is

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}.$$

Proof

If t_0 is a point in (α, β), the solution to this equation is

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}.$$

It follows that if $W(t_0) = 0$, then $W(t) = 0$ for all t in (α, β),

NTNU
Norwegian University of
Science and Technology

Proof

If t_0 is a point in (α, β), the solution to this equation is

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}.$$

It follows that if $W(t_0) = 0$, then $W(t) = 0$ for all t in (α, β), and if $W(t_0) \neq 0$, then $W(t) \neq 0$ for all t in (α, β).

Linear dependent functions

NTNU
Norwegian University of
Science and Technology

Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.

Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.
- u and v are *linear independent* if they are not linear dependent.

NTNU
Norwegian University of Science and Technology

Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.
- u and v are *linear independent* if they are not linear dependent.
- If $u = cv$ for some constant c, then u and v are linear dependent because $u - cv = 0$.

NTNU
Norwegian University of Science and Technology

Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.
- u and v are *linear independent* if they are not linear dependent.
- If $u = cv$ for some constant c, then u and v are linear dependent because $u - cv = 0$.
- Conversely, if u and v are linear dependent, then $c_1 u + c_2 v = 0$ for some choice of constants c_1 and c_2 which are no both zero,

NTNU
Norwegian University of Science and Technology

Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.
- u and v are *linear independent* if they are not linear dependent.
- If $u = cv$ for some constant c, then u and v are linear dependent because $u - cv = 0$.
- Conversely, if u and v are linear dependent, then $c_1 u + c_2 v = 0$ for some choice of constants c_1 and c_2 which are no both zero, and then $u = -(c_2/c_1)v$ if $c_1 \neq 0$, and $v = -(c_1/c_2)u$ if $c_2 \neq 0$.

Linear dependent functions

Two functions u and v are linear dependent if and only if u is a constant multiple of v, or v is a constant multiple of u.

Proposition 1.27

NTNU
Norwegian University of
Science and Technology

Proposition 1.27

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β).

NTNU
Norwegian University of Science and Technology

Proposition 1.27

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β).
Then y_1 and y_2 are linearly dependent on (α, β) if and only if the Wronskian of y_1 and y_2 is identically equal to zero on the interval (α, β).

NTNU
Norwegian University of
Science and Technology

Proof

NTNU
Norwegian University of Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c,

NTNU
Norwegian University of
Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$

NTNU
Norwegian University of
Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$
Similarly, if $y_2(t) = cy_1(t)$ some constant c,

NTNU
Norwegian University of
Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0$.
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0$.

NTNU
Norwegian University of
Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0.$
Conversely, suppose that $W(t) = 0$ for all t in (α, β).

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0$.
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0$.
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If
$y_2(t) = 0$, then $y_2(t) = 0y_1(t)$.

NTNU
Norwegian University of Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0.$
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β),

NTNU
Norwegian University of Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0$.
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0$.
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β), then it follows from the continuity of y_2 that there is a subinterval (c, d) of (α, β) which contains t_0 such that $y_2(t) \neq 0$ for all t in (c, d).

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0.$
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β), then it follows from the continuity of y_2 that there is a subinterval (c, d) of (α, β) which contains t_0 such that $y_2(t) \neq 0$ for all t in (c, d). On this interval we have

$$\frac{d}{dt}(y_1(t)/y_2(t))$$

NTNU
Norwegian University of Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0$.
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0$.
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β), then it follows from the continuity of y_2 that there is a subinterval (c, d) of (α, β) which contains t_0 such that $y_2(t) \neq 0$ for all t in (c, d). On this interval we have

$$\frac{d}{dt}(y_1(t)/y_2(t)) = \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{(y_2(t))^2}$$

NTNU
Norwegian University of
Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0.$
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β), then it follows from the continuity of y_2 that there is a subinterval (c, d) of (α, β) which contains t_0 such that $y_2(t) \neq 0$ for all t in (c, d). On this interval we have

$$\frac{d}{dt}(y_1(t)/y_2(t)) = \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{(y_2(t))^2} = \frac{-W(t)}{(y_2(t))^2}$$

NTNU
Norwegian University of
Science and Technology

Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0$.
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0$.
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β), then it follows from the continuity of y_2 that there is a subinterval (c, d) of (α, β) which contains t_0 such that $y_2(t) \neq 0$ for all t in (c, d). On this interval we have

$$\frac{d}{dt}(y_1(t)/y_2(t)) = \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{(y_2(t))^2} = \frac{-W(t)}{(y_2(t))^2} = 0.$$

NTNU
Norwegian University of Science and Technology

When is the Wronskian zero?

Hence, on the interval (c, d), $y_1(t)/y_2(t)$ is equal to a constant c, and $y_1(t) = cy_2(t)$.

NTNU
Norwegian University of Science and Technology

When is the Wronskian zero?

Hence, on the interval (c, d), $y_1(t)/y_2(t)$ is equal to a constant c, and $y_1(t) = cy_2(t)$. In particular, $y_1(t_0) = cy_2(t_0)$ and $y_1'(t_0) = cy_2'(t_0)$.

NTNU
Norwegian University of Science and Technology

When is the Wronskian zero?

Hence, on the interval (c, d), $y_1(t)/y_2(t)$ is equal to a constant c, and $y_1(t) = cy_2(t)$. In particular, $y_1(t_0) = cy_2(t_0)$ and $y_1'(t_0) = cy_2'(t_0)$.
Since both $y_1(t)$ and $cy_2(t)$ are solutions to the initial value problem

$$y'' + p(t)y' + q(t)y = 0,\ y(t_0) = y_1(t_0),\ y'(t_0) = y_1'(t_0)$$

on (α, β),

NTNU
Norwegian University of Science and Technology

When is the Wronskian zero?

Hence, on the interval (c, d), $y_1(t)/y_2(t)$ is equal to a constant c, and $y_1(t) = cy_2(t)$. In particular, $y_1(t_0) = cy_2(t_0)$ and $y_1'(t_0) = cy_2'(t_0)$.
Since both $y_1(t)$ and $cy_2(t)$ are solutions to the initial value problem

$$y'' + p(t)y' + q(t)y = 0,\ y(t_0) = y_1(t_0),\ y'(t_0) = y_1'(t_0)$$

on (α, β), it follows that $y_1(t) = cy_2(t)$ on (α, β).

NTNU
Norwegian University of Science and Technology

Theorem 1.23

NTNU
Norwegian University of
Science and Technology

Theorem 1.23

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

on the interval (α, β). Then any solution to the equation is a linear combination of y_1 and y_2.

Proof

NTNU
Norwegian University of
Science and Technology

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other,

NTNU
Norwegian University of Science and Technology

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.

NTNU
Norwegian University of Science and Technology

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.
Let y be a solution to the equation.

NTNU
Norwegian University of Science and Technology

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.
Let y be a solution to the equation. Pick a point t_0 in (α, β) and let $a = y(t_0)$ and $b = y'(t_0)$.

NTNU
Norwegian University of Science and Technology

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.
Let y be a solution to the equation. Pick a point t_0 in (α, β) and let $a = y(t_0)$ and $b = y'(t_0)$.
Since $W(t_0) \neq 0$ there exist constants c_1 and c_2 such that $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.

NTNU
Norwegian University of Science and Technology

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.
Let y be a solution to the equation. Pick a point t_0 in (α, β) and let $a = y(t_0)$ and $b = y'(t_0)$.
Since $W(t_0) \neq 0$ there exist constants c_1 and c_2 such that $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.
Then both y and $c_1 y_1 + c_2 y_2$ are solutions to the initial value problem $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = a$, $y'(t_0) = b$ on the interval (α, β).

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.
Let y be a solution to the equation. Pick a point t_0 in (α, β) and let $a = y(t_0)$ and $b = y'(t_0)$.
Since $W(t_0) \neq 0$ there exist constants c_1 and c_2 such that $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.
Then both y and $c_1 y_1 + c_2 y_2$ are solutions to the initial value problem $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = a$, $y'(t_0) = b$ on the interval (α, β). It follows that $y = c_1 y_1 + c_2 y_2$.

NTNU
Norwegian University of Science and Technology

Fundamental set of solutions

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.

NTNU
Norwegian University of Science and Technology

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either

NTNU
Norwegian University of Science and Technology

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,

NTNU
Norwegian University of Science and Technology

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,
  2. or by showing that the Wronskian of y_1 and y_2 is not zero at any point.

NTNU
Norwegian University of Science and Technology

Second-order homogeneous linear differential equations with constant coefficients

NTNU
Norwegian University of Science and Technology

Second-order homogeneous linear differential equations with constant coefficients

A second-order homogeneous linear differential equation with constant coefficients is a differential equation with can be written on the form

$$y'' + py' + qy = 0$$

where p and q are constants.

Finding solutions to second-order homogeneous linear differential equations with constant coefficients

NTNU
Norwegian University of Science and Technology

Finding solutions to second-order homogeneous linear differential equations with constant coefficients

Inserting $y(t) = e^{\lambda t}$ into the differential equation, we obtain

NTNU
Norwegian University of Science and Technology

Finding solutions to second-order homogeneous linear differential equations with constant coefficients

Inserting $y(t) = e^{\lambda t}$ into the differential equation, we obtain

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t}.$$

NTNU
Norwegian University of Science and Technology

Finding solutions to second-order homogeneous linear differential equations with constant coefficients

Inserting $y(t) = e^{\lambda t}$ into the differential equation, we obtain

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t}.$$

Since $e^{\lambda t} \neq 0$, we have that $y(t) = e^{\lambda t}$ is a solution if and only if $\lambda^2 + p\lambda + q = 0$.

NTNU
Norwegian University of Science and Technology

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.

NTNU
Norwegian University of
Science and Technology

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$. A root of the characteristic polynomial is called a *characteristic root*.

NTNU
Norwegian University of Science and Technology

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$. A root of the characteristic polynomial is called a *characteristic root*.

The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

NTNU
Norwegian University of Science and Technology

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

1. $p^2 - 4q > 0$,

NTNU
Norwegian University of Science and Technology

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

1. $p^2 - 4q > 0$,
2. $p^2 - 4q < 0$,

NTNU
Norwegian University of Science and Technology

Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

1. $p^2 - 4q > 0$,
2. $p^2 - 4q < 0$,
3. $p^2 - 4q = 0$.

NTNU
Norwegian University of Science and Technology

Distinct real root

NTNU
Norwegian University of
Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.

NTNU
Norwegian University of
Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.

NTNU
Norwegian University of
Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

NTNU
Norwegian University of
Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \lambda_2 e^{\lambda_1 t} e^{\lambda_2 t} - \lambda_1 e^{\lambda_2 t} e^{\lambda_1 t}$$

NTNU
Norwegian University of
Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) = \lambda_2 e^{\lambda_1 t} e^{\lambda_2 t} - \lambda_1 e^{\lambda_2 t} e^{\lambda_1 t} \\ &= (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) t} \end{aligned}$$

NTNU
Norwegian University of Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$\begin{aligned} W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) &= \lambda_2 e^{\lambda_1 t} e^{\lambda_2 t} - \lambda_1 e^{\lambda_2 t} e^{\lambda_1 t} \\ &= (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t} \neq 0 \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) = \lambda_2 e^{\lambda_1 t} e^{\lambda_2 t} - \lambda_1 e^{\lambda_2 t} e^{\lambda_1 t} \\ &= (\lambda_2 - \lambda_1)e^{(\lambda_1 + \lambda_2)t} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants.

NTNU
Norwegian University of Science and Technology

Example

NTNU
Norwegian University of
Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 3\lambda + 2$,

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 3\lambda + 2$, and the characteristic roots are $\lambda = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 3\lambda + 2$, and the characteristic roots are $\lambda = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$.
It follows that the general solution is $y(t) = c_1 e^{2t} + c_2 e^{t}$.

NTNU
Norwegian University of Science and Technology

Complex roots

NTNU
Norwegian University of
Science and Technology

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2

NTNU
Norwegian University of Science and Technology

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2 which have the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ where a and b are real numbers and $b \neq 0$.

NTNU
Norwegian University of Science and Technology

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2 which have the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ where a and b are real numbers and $b \neq 0$.
The 2 functions $z_1(t) = e^{\lambda_1 t} = e^{at}(\cos(bt) + i\sin(bt))$ and $z_2(t) = e^{\lambda_2 t} = e^{at}(\cos(bt) - i\sin(bt))$ are then solutions to the differential equation $y'' + py' + qy = 0$.

NTNU
Norwegian University of Science and Technology

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2 which have the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ where a and b are real numbers and $b \neq 0$.
The 2 functions $z_1(t) = e^{\lambda_1 t} = e^{at}(\cos(bt) + i\sin(bt))$ and $z_2(t) = e^{\lambda_2 t} = e^{at}(\cos(bt) - i\sin(bt))$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The two functions $y_1(t) = \mathrm{Re}(z_1(t)) = e^{at}\cos(bt)$ and $y_2(t) = \mathrm{Im}(z_1(t)) = e^{at}\sin(bt)$ are also solutions to the differential equation $y'' + py' + qy = 0$.

NTNU
Norwegian University of Science and Technology

Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

NTNU
Norwegian University of Science and Technology

Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$\begin{aligned} W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ =& e^{at}\cos(bt)(ae^{at}\sin(bt) + be^{at}\cos(bt)) \\ & - e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt)) \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$\begin{aligned}W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t)\\ =& e^{at}\cos(bt)(ae^{at}\sin(bt) + be^{at}\cos(bt))\\ &- e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt))\\ =& be^{2at}\cos^2(bt) + be^{2at}\sin^2(bt)\end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$\begin{aligned}
W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\
=& e^{at}\cos(bt)(ae^{at}\sin(bt) + be^{at}\cos(bt)) \\
& - e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt)) \\
=& be^{2at}\cos^2(bt) + be^{2at}\sin^2(bt) \\
=& be^{2at} \neq 0
\end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$\begin{aligned}W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t)\\ =& e^{at}\cos(bt)(ae^{at}\sin(bt) + be^{at}\cos(bt))\\ &- e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt))\\ =& be^{2at}\cos^2(bt) + be^{2at}\sin^2(bt)\\ =& be^{2at} \neq 0\end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$ where c_1 and c_2 are constants.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' + 2y' + 2y = 0$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' + 2y' + 2y = 0$.
The characteristic polynomial is $\lambda^2 + 2\lambda + 2$,

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' + 2y' + 2y = 0$.
The characteristic polynomial is $\lambda^2 + 2\lambda + 2$, and the characteristic roots are
$\lambda = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm-4}{2} = \frac{-2\pm 2i}{2} = -1 \pm i$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' + 2y' + 2y = 0$.
The characteristic polynomial is $\lambda^2 + 2\lambda + 2$, and the characteristic roots are
$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm -4}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$.
It follows that the general solution is
$y(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$.

NTNU
Norwegian University of
Science and Technology

Repeated roots

NTNU
Norwegian University of
Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.

NTNU
Norwegian University of Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.

NTNU
Norwegian University of Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$.

NTNU
Norwegian University of Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,

NTNU
Norwegian University of
Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$

NTNU
Norwegian University of
Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$y_2''(t) + py_2'(t) + qy_2(t)$$

NTNU
Norwegian University of Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$y_2''(t) + py_2'(t) + qy_2(t) = 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + p(e^{\lambda t} + \lambda te^{\lambda t}) + qte^{\lambda t}$$

NTNU
Norwegian University of
Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$\begin{aligned} y_2''(t) + py_2'(t) + qy_2(t) &= 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + p(e^{\lambda t} + \lambda te^{\lambda t}) + qte^{\lambda t} \\ &= (2\lambda + p)e^{\lambda t} + (\lambda^2 + p\lambda + q)te^{\lambda t} \end{aligned}$$

NTNU
Norwegian University of Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$\begin{aligned} y_2''(t) + py_2'(t) + qy_2(t) &= 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + p(e^{\lambda t} + \lambda te^{\lambda t}) + qte^{\lambda t} \\ &= (2\lambda + p)e^{\lambda t} + (\lambda^2 + p\lambda + q)te^{\lambda t} \\ &= 0. \end{aligned}$$

NTNU
Norwegian University of Science and Technology

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$\begin{aligned} y_2''(t) + py_2'(t) + qy_2(t) &= 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + p(e^{\lambda t} + \lambda te^{\lambda t}) + qte^{\lambda t} \\ &= (2\lambda + p)e^{\lambda t} + (\lambda^2 + p\lambda + q)te^{\lambda t} \\ &= 0. \end{aligned}$$

So y_2 is a solution to the differential equation $y'' + py' + qy = 0$.

NTNU
Norwegian University of Science and Technology

Repeated roots

The Wronskian of $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ is

$$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

NTNU
Norwegian University of
Science and Technology

Repeated roots

The Wronskian of $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ is

$$\begin{aligned} W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ =& e^{\lambda t}(e^{\lambda t} + \lambda t e^{\lambda t}) - \lambda e^{\lambda t} t e^{\lambda t} \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Repeated roots

The Wronskian of $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ is

$$\begin{aligned} W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ =& e^{\lambda t}(e^{\lambda t} + \lambda t e^{\lambda t}) - \lambda e^{\lambda t} t e^{\lambda t} \\ =& e^{2\lambda t} \neq 0 \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Repeated roots

The Wronskian of $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ is

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ &= e^{\lambda t}(e^{\lambda t} + \lambda te^{\lambda t}) - \lambda e^{\lambda t}te^{\lambda t} \\ &= e^{2\lambda t} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 e^{\lambda t} + c_2 te^{\lambda t}$ where c_1 and c_2 are constants.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 2\lambda + 1$,

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 2\lambda + 1$, and the characteristic roots are $\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 2\lambda + 1$, and the characteristic roots are $\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$.
It follows that the general solution is $y(t) = c_1 e^t + c_2 t e^t$.

NTNU
Norwegian University of Science and Technology

Exercise 4.3.26

Find the solution to the initial value problem

$$10y'' - y' - 3y = 0, \; y(0) = 1, \; y'(0) = 0.$$

NTNU
Norwegian University of
Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$,

NTNU
Norwegian University of
Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

NTNU
Norwegian University of
Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.

NTNU
Norwegian University of Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.
Let $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.

NTNU
Norwegian University of Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.

Let $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$. Then

$y'(t) = (3/5)c_1 e^{3t/5} - (c_2/2)e^{-t/2}$,

NTNU
Norwegian University of Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.
Let $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$. Then
$y'(t) = (3/5)c_1 e^{3t/5} - (c_2/2)e^{-t/2}$, $y'(0) = 3c_1/5 - c_2/2$,

NTNU
Norwegian University of Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.
Let $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$. Then $y'(t) = (3/5)c_1 e^{3t/5} - (c_2/2)e^{-t/2}$, $y'(0) = 3c_1/5 - c_2/2$, and $y(0) = c_1 + c_2$.

NTNU
Norwegian University of Science and Technology

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1\pm\sqrt{1+120}}{20} = \frac{1\pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.

Let $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$. Then $y'(t) = (3/5)c_1 e^{3t/5} - (c_2/2)e^{-t/2}$, $y'(0) = 3c_1/5 - c_2/2$, and $y(0) = c_1 + c_2$.

So y is a solution to the initial value problem if and only if $c_1 + c_2 = 1$ and $3c_1/5 - c_2/2 = 0$.

NTNU
Norwegian University of Science and Technology

Solution

The solution to the linear system

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1/5 - c_2/2 &= 0 \end{aligned}$$

is $c_1 = 5/11$ and $c_2 = 6/11$,

NTNU
Norwegian University of
Science and Technology

Solution

The solution to the linear system

$$c_1 + c_2 = 1$$
$$3c_1/5 - c_2/2 = 0$$

is $c_1 = 5/11$ and $c_2 = 6/11$, so the solution to the initial value problem

$$10y'' - y' - 3y = 0, \; y(0) = 1, \; y'(0) = 0.$$

is $y(t) = (5/11)e^{3t/5} + (6/11)e^{-t/2}$.

NTNU
Norwegian University of Science and Technology

Plan for next week

Plan for next week

Wednesday we shall

NTNU
Norwegian University of
Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,

NTNU
Norwegian University of
Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,

NTNU
Norwegian University of Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).

NTNU
Norwegian University of Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).
Thursday we shall

NTNU
Norwegian University of Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).
Thursday we shall

- look at *variation of parameters*,

NTNU
Norwegian University of Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).

Thursday we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

NTNU
Norwegian University of
Science and Technology

Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).

Thursday we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

Section 4.6 and 4.7 (pages lxxii–lxxxvi).

NTNU
Norwegian University of
Science and Technology