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TMA4115 - Calculus 3

Lecture 28, April 25

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday’s lecture

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Yesterday's lecture

Yesterday we looked at

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Yesterday's lecture

Yesterday we looked at

- *least-squares problems*,

Yesterday's lecture

Yesterday we looked at

- *least-squares problems*,
- applications to linear models.

Today's lecture

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Today's lecture

Today we shall introduce and study

Today's lecture

Today we shall introduce and study

- *symmetric matrices*,

Today's lecture

Today we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

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Symmetric matrices

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Symmetric matrices

- A *symmetric* matrix is a matrix A such that $A^T = A$.

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Symmetric matrices

- A *symmetric* matrix is a matrix A such that $A^T = A$.
- A symmetric matrix is necessarily square.
- We will in this lecture see that every symmetric matrix is orthogonally diagonalizable, that is, if A is symmetric, then $A = PDP^{-1}$ where P is an orthogonal matrix and D is a diagonal matrix.

Examples of symmetric matrices

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -3 \\ 1 & 2 & 4 \\ -3 & 4 & 7 \end{bmatrix}$$

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Orthogonal matrices

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Recall that an *orthogonal matrix* is a square matrix whose columns form an orthonormal set.

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Let P be an $n \times n$ matrix. Then the following statements are logically equivalent:

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4. P is invertible and $P^{-1} = P^T$.

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3. $P^T P = I_n$.
4. P is invertible and $P^{-1} = P^T$.
5. $PP^T = I_n$.
6. The rows of P form an orthonormal basis for $\mathbb{R}^n$.

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Recall that an *orthogonal matrix* is a square matrix whose columns form an orthonormal set.

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Let P be an $n \times n$ matrix. Then the following statements are logically equivalent:

1. P is an orthogonal matrix.
2. The columns of P form an orthonormal basis for $\mathbb{R}^n$.
3. $P^T P = I_n$.
4. P is invertible and $P^{-1} = P^T$.
5. $PP^T = I_n$.
6. The rows of P form an orthonormal basis for $\mathbb{R}^n$.
7. P^T is an orthogonal matrix.

Example

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Example

Let $P = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}$.

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Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set,

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Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set, so P is an orthogonal matrix.

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Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set, so P is an orthogonal matrix. Notice also that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

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We have that

$$P^T P = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \mathbf{u}_1 \cdot \mathbf{u}_3 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \mathbf{u}_2 \cdot \mathbf{u}_3 \\ \mathbf{u}_3 \cdot \mathbf{u}_1 & \mathbf{u}_3 \cdot \mathbf{u}_2 & \mathbf{u}_3 \cdot \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

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It follows (from the inverse matrix theorem) that P is invertible and $P^{-1} = P^T$.

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It follows (from the inverse matrix theorem) that P is invertible and $P^{-1} = P^T$. So $P^T P = I_3$, which means that the rows of P forms an orthonormal basis for $\mathbb{R}^3$ and that P^T is orthogonal.

Orthogonally diagonalization

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Orthogonally diagonalization

An $n \times n$ matrix A is *orthogonally diagonalizable* if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

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An $n \times n$ matrix A is *orthogonally diagonalizable* if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Theorem 2

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric.

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Example

Let $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Let us find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution

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The eigenvalues of A are -2 and 7.

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$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ form a basis for the eigenspace of A corresponding to 7.

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$\mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to -2.

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Notice that $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

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Let $\mathbf{z}_1 = \mathbf{v}_1$

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Notice that $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

Let $\mathbf{z}_1 = \mathbf{v}_1$ and $\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{z}_1}{\mathbf{z}_1 \cdot \mathbf{z}_1}\mathbf{z}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix}$.

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Solution (cont.)

And let $\mathbf{z}_3 = \mathbf{v}_3$.

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And let $\mathbf{z}_3 = \mathbf{v}_3$. Then $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is an orthogonal set.

Solution (cont.)

And let $\mathbf{z}_3 = \mathbf{v}_3$. Then $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is an orthogonal set. Let

$$\mathbf{x}_1 = \frac{1}{\|\mathbf{z}_1\|}\mathbf{z}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\mathbf{x}_2 = \frac{1}{\|\mathbf{z}_2\|}\mathbf{z}_2 = \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \text{ and}$$

$$\mathbf{x}_3 = \frac{1}{\|\mathbf{z}_3\|}\mathbf{z}_3 = \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

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Solution (cont.)

Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal set,

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Solution (cont.)

Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal set, so if we let

$$P = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix},$$

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Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal set, so if we let $P = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}$, then P is an orthogonal matrix, and if we let $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then $A = PDP^{-1}$.

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Orthogonal eigenvectors

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Orthogonal eigenvectors

Theorem 1

If A is a symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.

Proof of Theorem 1

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Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively, and assume that $\lambda_1 \neq \lambda_2$.

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Proof of Theorem 1

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively, and assume that $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned}\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 &= (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 \\ &= \mathbf{v}_1^T A \mathbf{v}_2 = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \mathbf{v}_1 \cdot (\lambda_2 \mathbf{v}_2) = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2\end{aligned}$$

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Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively, and assume that $\lambda_1 \neq \lambda_2$. Then

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so $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

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Proof of Theorem 1

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively, and assume that $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned}\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 &= (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 \\ &= \mathbf{v}_1^T A \mathbf{v}_2 = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \mathbf{v}_1 \cdot (\lambda_2 \mathbf{v}_2) = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2\end{aligned}$$

so $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Since $\lambda_1 \neq \lambda_2$, it follows that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

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The spectral theorem for symmetric matrices

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The spectral theorem for symmetric matrices

Theorem 3

An $n \times n$ symmetric matrix A has the following properties:

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The spectral theorem for symmetric matrices

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The spectral theorem for symmetric matrices

Theorem 3

An $n \times n$ symmetric matrix A has the following properties:

1. A has n real eigenvalues, counting multiplicities.
2. The dimensions of the eigenspace for each eigenvalue λ equals the multiplicity of λ.
3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

Spectral decomposition

Spectral decomposition

Suppose $A = PDP^{-1}$ where $P = [\mathbf{u}_1 \dots \mathbf{u}_n]$ is an orthogonal matrix and $D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$.

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If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$ and $A\mathbf{x} = \lambda_1(\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + \lambda_n(\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$.

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We furthermore have that

$$A = \lambda \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

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We furthermore have that

$$A = \lambda \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

This is called a *spectral decomposition* of A.

Example

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Example

If $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$,

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Example

If $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, then

$$A = 7 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} + 7 \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{bmatrix}$$

$$- 2 \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= 7 \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + 7 \begin{bmatrix} \frac{1}{18} & \frac{-4}{18} & \frac{-1}{18} \\ \frac{-4}{18} & \frac{16}{18} & \frac{4}{18} \\ \frac{-1}{18} & \frac{4}{18} & \frac{1}{18} \end{bmatrix} - 2 \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{-2}{9} & \frac{4}{9} \end{bmatrix}$$

Quadratic forms

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Quadratic forms

- A *quadratic form* on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric $n \times n$ matrix A.

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- The matrix A is called the *matrix of Q*.
- Quadratic forms occupy a central place in various branches of mathematics, including
  - number theory,
  - linear algebra,
  - group theory,
  - differential geometry,
  - differential topology,
  - Lie theory.

Examples of quadratic forms

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Examples of quadratic forms

- $\mathbf{x} \to \|\mathbf{x}\|^2$ is a quadratic form because $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x}^T I_n \mathbf{x}$.

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- $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$ is a quadratic from because $6x_1^2 - 24x_1x_2 - x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

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Examples of quadratic forms

- $\mathbf{x} \to \|\mathbf{x}\|^2$ is a quadratic form because $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x}^T I_n \mathbf{x}$.
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  Let $y_1 = \frac{4}{5}x_1 - \frac{3}{5}x_2$ and $y_2 = \frac{3}{5}x_1 + \frac{4}{5}x_2$. Then $6x_1^2 - 24x_1x_2 - x_2^2 = 15y_1^2 - 10y_2^2$.

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The principal axes theorem

Theorem 4

Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product term.

The principal axes theorem

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The above theorem can be used to classify conic sections.

Example

Example

Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

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Example

Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

The matrix of this quadratic form is $A = \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix}$.

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Example

Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

The matrix of this quadratic form is $A = \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix}$.

$\det(A - \lambda I_2) = \begin{vmatrix} 6-\lambda & -12 \\ -12 & -1-\lambda \end{vmatrix} = (6-\lambda)(-1-\lambda) - 144 =$
$\lambda^2 - 5\lambda - 150.$

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Example

Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

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$$\lambda = \frac{5 \pm \sqrt{5^2 - 4(-150)}}{2} = \frac{5 \pm \sqrt{625}}{2} = \frac{5 \pm 25}{2} = \begin{cases} 15 \\ -10 \end{cases},$$

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Example

Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

The matrix of this quadratic form is $A = \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix}$.

$\det(A - \lambda I_2) = \begin{vmatrix} 6-\lambda & -12 \\ -12 & -1-\lambda \end{vmatrix} = (6-\lambda)(-1-\lambda) - 144 =$ $\lambda^2 - 5\lambda - 150$. and the zeros of $\lambda^2 - 5\lambda - 150$ are $\lambda = \frac{5 \pm \sqrt{5^2 - 4(-150)}}{2} = \frac{5 \pm \sqrt{625}}{2} = \frac{5 \pm 25}{2} = \begin{cases} 15 \\ -10 \end{cases}$, so the eigenvalues of A are 15 and -10.

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Example (cont.)

$A - 15I_2 = \begin{bmatrix} -9 & -12 \\ -12 & -16 \end{bmatrix}$ so $\mathbf{v}_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue 15.

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Example (cont.)

$A - 15I_2 = \begin{bmatrix} -9 & -12 \\ -12 & -16 \end{bmatrix}$ so $\mathbf{v}_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue 15.

$A + 10I_2 = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}$ so $\mathbf{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue -10.

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Example (cont.)

$A - 15I_2 = \begin{bmatrix} -9 & -12 \\ -12 & -16 \end{bmatrix}$ so $\mathbf{v}_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue 15.

$A + 10I_2 = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}$ so $\mathbf{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue -10.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 5$,

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Example (cont.)

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$A + 10I_2 = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}$ so $\mathbf{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue -10.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 5$, so if we let $P = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix}$, then P is an orthogonal matrix and $A = PDP^{-1}$.

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Example (cont.)

Let $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$
$\begin{bmatrix} \frac{4}{5}x_1 + \frac{3}{5}x_2 \\ \frac{-3}{5}x_1 + \frac{4}{5}x_2 \end{bmatrix}$.

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Example (cont.)

Let $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$
$\begin{bmatrix} \frac{4}{5}x_1 + \frac{3}{5}x_2 \\ \frac{-3}{5}x_1 + \frac{4}{5}x_2 \end{bmatrix}$.
Then

$$\begin{aligned} 6x_1^2 - 24x_1x_2 - x_2^2 &= [x_1 \ x_2]A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2]P^{-1}DP \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= [y_1 \ y_2]D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [y_1 \ y_2] \begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= 15y_1^2 - 10y_2^2 \end{aligned}$$

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Example (cont.)

The set of points (y_1, y_2) satisfying $15y_1^2 - 10y_2^2 = 1$ is a hyperbola.

Example (cont.)

The set of points (y_1, y_2) satisfying $15y_1^2 - 10y_2^2 = 1$ is a hyperbola. It follows that the set of points (x_1, x_2) satisfying $6x_1^2 - 24x_1x_2 - x_2^2 = 1$ is a rotated hyperbola.

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Classifying quadratic forms

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Classifying quadratic forms

Definition

A quadratic form Q is said to be:

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Classifying quadratic forms

Definition

A quadratic form Q is said to be:

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,

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- *indefinite* if $Q(\mathbf{x})$ assumes both positive and negative values.

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- *indefinite* if $Q(\mathbf{x})$ assumes both positive and negative values.

- If Q is a positive definite quadratic form on $\mathbb{R}^2$, then set of points (x_1, x_2) satisfying $Q(x_1, x_2) = 1$ forms an ellipse.

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- If Q is a positive definite quadratic form on $\mathbb{R}^2$, then set of points (x_1, x_2) satisfying $Q(x_1, x_2) = 1$ forms an ellipse.
- If Q is an indefinite quadratic form on $\mathbb{R}^2$, then set of points (x_1, x_2) satisfying $Q(x_1, x_2) = 1$ forms a hyperbola.

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Quadratic forms and eigenvalues

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Quadratic forms and eigenvalues

Theorem 5

Let A be an $n \times n$ symmetric matrix, and let Q be the quadratic form $x \mapsto \mathbf{x}^t A\mathbf{x}$.

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Quadratic forms and eigenvalues

Theorem 5

Let A be an $n \times n$ symmetric matrix, and let Q be the quadratic form $x \mapsto \mathbf{x}^t A\mathbf{x}$.

1. Q is positive definite if and only if the eigenvalues of A are all positive.

Quadratic forms and eigenvalues

Theorem 5

Let A be an $n \times n$ symmetric matrix, and let Q be the quadratic form $x \mapsto \mathbf{x}^t A\mathbf{x}$.

1. Q is positive definite if and only if the eigenvalues of A are all positive.
2. Q is negative definite if and only if the eigenvalues of A are all negative.

Quadratic forms and eigenvalues

Theorem 5

Let A be an $n \times n$ symmetric matrix, and let Q be the quadratic form $x \mapsto \mathbf{x}^t A\mathbf{x}$.

1. Q is positive definite if and only if the eigenvalues of A are all positive.
2. Q is negative definite if and only if the eigenvalues of A are all negative.
3. Q is indefinite if and only if A has both positive and negative eigenvalues.

Problem 7 from June 2010

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of A.
2. Find a matrix P and a diagonal matrix D such that $A = PDP^T$.
3. Solve the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 + y_2 + y_3 \\ y_2' &= y_1 + 2y_2 \\ y_3' &= y_1 + 2y_3 \end{aligned}$$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$.

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Solution

Solution

$$\det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix} + (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= -(2-\lambda) + (2-\lambda)\big((3-\lambda)(2-\lambda) - 1\big)$$

$$= (2-\lambda)\big((3-\lambda)(2-\lambda) - 2\big)$$

$$= (2-\lambda)(\lambda^2 - 5\lambda - 4)$$

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Solution

$$\det(A-\lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

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$$= -(2-\lambda) + (2-\lambda)\big((3-\lambda)(2-\lambda)-1\big)$$

$$= (2-\lambda)\big((3-\lambda)(2-\lambda)-2\big)$$

$$= (2-\lambda)(\lambda^2-5\lambda-4)$$

and $\lambda^2-5\lambda-4=0$ if and only if

$$\lambda = \frac{5\pm\sqrt{5^2-4(-4)}}{2} = \frac{5\pm 3^2}{2} = \begin{cases} 4 \\ 1 \end{cases},$$

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Solution (cont.)

so the eigenvalues of A are 1,2 and 4.

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Solution (cont.)

so the eigenvalues of A are 1,2 and 4.
To find the eigenvectors of A corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.

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Solution (cont.)

so the eigenvalues of A are 1,2 and 4.
To find the eigenvectors of A corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.

$$A - I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

so the eigenvalues of A are 1,2 and 4.
To find the eigenvectors of A corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.

$$A - I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set of eigenvectors of A corresponding to 1 is $\left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.

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Solution (cont.)

$$A - 2I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution (cont.)

$A - 2I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ so the set of eigenvectors of A corresponding to 2 is $\left\{ t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : t \neq 0 \right\}$.

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Solution (cont.)

To find the eigenvectors of A corresponding to 4, we reduce $A - 4I_3$ to its reduced echelon form.

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Solution (cont.)

To find the eigenvectors of A corresponding to 4, we reduce $A - 4I_3$ to its reduced echelon form.

$$A - 4I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution (cont.)

To find the eigenvectors of A corresponding to 4, we reduce $A - 4I_3$ to its reduced echelon form.

$$A - 4I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set of eigenvectors of A corresponding to 4 is $\left\{ t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.

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Solution (cont.)

To find a matrix P and a diagonal matrix D such that $A = PDP^T$ we will orthogonally diagonalize A.

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Solution (cont.)

To find a matrix P and a diagonal matrix D such that $A = PDP^T$ we will orthogonally diagonalize A. For that we need an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

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Solution (cont.)

To find a matrix P and a diagonal matrix D such that $A = PDP^T$ we will orthogonally diagonalize A. For that we need an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Since A is symmetric, eigenvectors of A corresponding to different eigenvalues are orthogonal to each other, so we just have to find a unit vector in each of the 3 eigenspaces of A.

Solution (cont.)

We have that $\begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector of A corresponding to 1, that $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ is a unit eigenvector of A corresponding to 2, and that $\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector of A corresponding to 4.

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Solution (cont.)

So if we let $P = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

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Solution (cont.)

So if we let $P = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A = PDP^{-1} = PDP^T$.

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Solution (cont.)

The coefficient matrix the system

$$\begin{aligned} y_1' &= 3y_1 + y_2 + y_3 \\ y_2' &= y_1 + 2y_2 \\ y_3' &= y_1 + 2y_3 \end{aligned}$$

is A,

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Solution (cont.)

so since the eigenvalues of A are 1,2 and 4, and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 1, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to 2, and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 4,

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Solution (cont.)

so since the eigenvalues of A are 1,2 and 4, and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 1, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to 2, and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 4, the general solution of the system is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t}.$$

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Solution (cont.)

If $\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t}$, then

$$\begin{aligned} \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} &= c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

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Solution (cont.)

so $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$ if and only if

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

Solution (cont.)

so $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$ if and only if

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

We reduce the augmented matrix of the above system to its reduced echelon form.

Solution (cont.)

$$\begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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Solution (cont.)

$$\begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We see that the solution to the system

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

is $c_1 = -1, c_2 = 2, c_3 = 1.$

Solution (cont.)

So the solution to the system of differential equations

$$
\begin{aligned}
y_1' &= 3y_1 + y_2 + y_3 \\
y_2' &= y_1 + 2y_2 \\
y_3' &= y_1 + 2y_3
\end{aligned}
$$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$, is

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} 2e^{4t} + e^t \\ e^{4t} + 2e^{2t} - e^t \\ e^{4t} - 2e^{2t} - e^t \end{bmatrix}
$$