

TMA4115 - Calculus 3 Lecture 28, April 25

Toke Meier Carlsen Norwegian University of Science and Technology Spring 2013



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Yesterday we looked at



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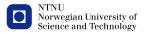
Yesterday we looked at

• least-squares problems,



Yesterday we looked at

- least-squares problems,
- applications to linear models.





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Today we shall introduce and study



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• symmetric matrices,



Today we shall introduce and study

- symmetric matrices,
- quadratic forms.





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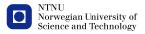
• A symmetric matrix is a matrix A such that $A^T = A$.



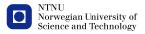
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- A symmetric matrix is a matrix A such that $A^T = A$.
- A symmetric matrix is necessarily square.
- We will in this lecture see that every symmetric matrix is orthogonally diagonalizable, that is, if A is symmetric, then A = PDP⁻¹ where P is an orthogonal matrix and D is a diagonal matrix.



Examples of symmetric matrices

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & -3 \\ 1 & 2 & 4 \\ -3 & 4 & 7 \end{bmatrix}$$





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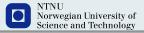
Recall that an *orthogonal matrix* is a square matrix whose columns form an orthonormal set.



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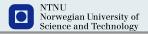


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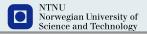
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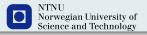
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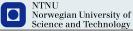
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- *P* is an orthogonal matrix.
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- $P^T P = I_n.$
- *P* is invertible and $P^{-1} = P^T$.

$$PP^{T} = I_{n}.$$

- **•** The rows of *P* form an orthonormal basis for \mathbb{R}^n .
- P^{T} is an orthogonal matrix.





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Let
$$P = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{vmatrix}$$
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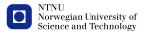
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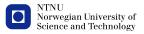
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Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set, so *P* is an orthogonal matrix.



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Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set, so *P* is an orthogonal matrix. Notice also that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for \mathbb{R}^3 .

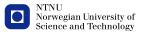


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We have that

$$P^{T}P = \begin{bmatrix} \mathbf{u}_{1} \cdot \mathbf{u}_{1} & \mathbf{u}_{1} \cdot \mathbf{u}_{2} & \mathbf{u}_{1} \cdot \mathbf{u}_{3} \\ \mathbf{u}_{2} \cdot \mathbf{u}_{1} & \mathbf{u}_{2} \cdot \mathbf{u}_{2} & \mathbf{u}_{2} \cdot \mathbf{u}_{3} \\ \mathbf{u}_{3} \cdot \mathbf{u}_{1} & \mathbf{u}_{3} \cdot \mathbf{u}_{2} & \mathbf{u}_{3} \cdot \mathbf{u}_{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3}.$$

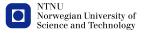


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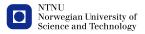
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Orthogonally diagonalization

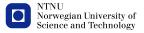


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Orthogonally diagonalization

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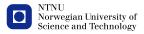
Theorem 2

An $n \times n$ matrix *A* is orthogonally diagonalizable if and only if *A* is symmetric.



Example

Let $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. Let us find an orthogonal matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.

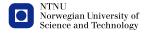




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The eigenvalues of A are -2 and 7.



The eigenvalues of *A* are
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 $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ form a basis for the eigenspace of *A* corresponding to 7.



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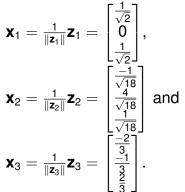
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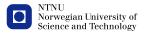
$$\boldsymbol{P} = [\mathbf{X}_1 \ \mathbf{X}_2 \ \mathbf{X}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix},$$



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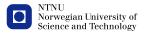
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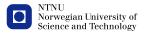
matrix, and if we let $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix},$



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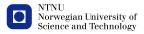
Orthogonal eigenvectors



Orthogonal eigenvectors

Theorem 1

If *A* is a symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.





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Let \mathbf{v}_1 and \mathbf{v}_2 be eigenvectors of A corresponding to λ_1 and λ_2 , respectively, and assume that $\lambda_1 \neq \lambda_2$.



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$$= \mathbf{v}_1^T \mathbf{A} \mathbf{v}_2 = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \mathbf{v}_1 \cdot (\lambda_2 \mathbf{v}_2) = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2$$



Let \mathbf{v}_1 and \mathbf{v}_2 be eigenvectors of A corresponding to λ_1 and λ_2 , respectively, and assume that $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = (A \mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2$$
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so
$$(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{0}$$
.

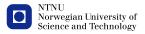


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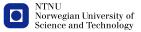
so $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Since $\lambda_1 \neq \lambda_2$, it follows that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.





Theorem 3

An $n \times n$ symmetric matrix A has the following properties:



Theorem 3

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- A has n real eigenvalues, counting multiplicities.
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- The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.



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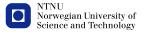
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Suppose $A = PDP^{-1}$ where $P = [\mathbf{u}_1 \dots \mathbf{u}_n]$ is an orthogonal matrix and $D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$.



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$$\mathbf{A} = \lambda \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$



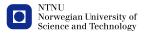
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$$\mathbf{A} = \lambda \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

This is called a *spectral decomposition* of *A*.

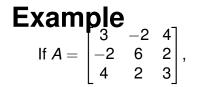


Example



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Example
If
$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$
, then

$$A = 7 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} + 7 \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{bmatrix}$$

$$-2 \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$= 7 \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + 7 \begin{bmatrix} \frac{1}{18} & \frac{-4}{18} & \frac{-1}{18} \\ \frac{-4}{18} & \frac{1}{18} & \frac{4}{18} \end{bmatrix} - 2 \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{9}{2} & \frac{1}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{1}{2} & \frac{-2}{9} \end{bmatrix}$$
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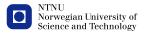
A quadratic form on ℝⁿ is a function Q defined on ℝⁿ such that Q(x) = x^TAx for some symmetric n × n matrix A.



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- A quadratic form on ℝⁿ is a function Q defined on ℝⁿ such that Q(x) = x^TAx for some symmetric n × n matrix A.
- The matrix A is called the *matrix of Q*.
- Quadratic forms occupy a central place in various branches of mathematics, including
 - number theory,
 - linear algebra,
 - group theory,
 - differential geometry,
 - differential topology,
 - Lie theory.





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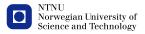
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• $\mathbf{x} \to \|\mathbf{x}\|^2$ is a quadratic form because $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T \mathbf{x} = \mathbf{x}^T I_n \mathbf{x}.$



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• $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$ is a quadratic from because $6x_1^2 - 24x_1x_2 - x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$



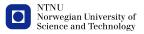
• $\mathbf{x} \to \|\mathbf{x}\|^2$ is a quadratic form because $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T \mathbf{x} = \mathbf{x}^T I_n \mathbf{x}.$ • $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$ is a quadratic from because $6x_1^2 - 24x_1x_2 - x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$ Let $y_1 = \frac{4}{5}x_1 - \frac{3}{5}x_2$ and $y_2 = \frac{3}{5}x_1 + \frac{4}{5}x_2$. Then $6x_1^2 - 24x_1x_2 - x_2^2 = 15y_1^2 - 10y_2^2.$



The principal axes theorem

Theorem 4

Let *A* be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A \mathbf{x}$ into a quadratic form $\mathbf{y}^T D \mathbf{y}$ with no cross-product term.



The principal axes theorem

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The above theorem can be used to classify conic sections.





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Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

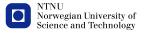


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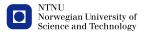


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$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \begin{vmatrix} \mathbf{6} - \lambda & -\mathbf{12} \\ -\mathbf{12} & -\mathbf{1} - \lambda \end{vmatrix} = (\mathbf{6} - \lambda)(-\mathbf{1} - \lambda) - \mathbf{144} = \lambda^2 - 5\lambda - \mathbf{150}.$$

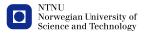


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eigenvalues of A are 15 and -10.



$$A - 15I_2 = \begin{bmatrix} -9 & -12 \\ -12 & -16 \end{bmatrix}$$
 so $\mathbf{v}_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue 15.



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$$A + 10I_2 = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \text{ so } \mathbf{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ is an eigenvalue of } A$$

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$$\boldsymbol{v}_1\cdot\boldsymbol{v}_2=0 \text{ and } \|\boldsymbol{v}_1\|=\|\boldsymbol{v}_2\|=5,$$



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corresponding to the eigenvalue -10.
$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \text{ and } \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 5, \text{ so if we let } P = \frac{1}{5} \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$$

and
$$D = \begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix}, \text{ then } P \text{ is an orthogonal matrix and}$$

$$A = PDP^{-1}.$$



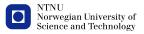
Let
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



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$$\begin{bmatrix} \frac{4}{5} x_1 + \frac{3}{5} x_2 \\ \frac{-3}{5} x_1 + \frac{4}{5} x_2 \end{bmatrix}.$$
Then

$$6x_1^2 - 24x_1x_2 - x_2^2 = [x_1 \ x_2]A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2]P^{-1}DP \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= [y_1 \ y_2]D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [y_1 \ y_2] \begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= 15y_1^2 - 10y_2^2$$

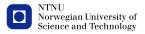


The set of points (y_1, y_2) satisfying $15y_1^2 - 10y_2^2 = 1$ is a hyperbola.



The set of points (y_1, y_2) satisfying $15y_1^2 - 10y_2^2 = 1$ is a hyperbola. It follows that the set of points (x_1, x_2) satisfying $6x_1^2 - 24x_1x_2 - x_2^2 = 1$ is a rotated hyperbola.





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Definition



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A quadratic form *Q* is said to be:

• positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,



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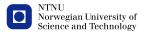
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- indefinite if Q(x) assumes both positive and negative values.
- If Q is a positive definite quadratic form on ℝ², then set of points (x₁, x₂) satisfying Q(x₁, x₂) = 1 forms an ellipse.
- If Q is an indefinite quadratic form on ℝ², then set of points (x₁, x₂) satisfying Q(x₁, x₂) = 1 forms a hyperbola.



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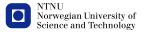


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Theorem 5

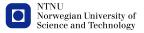
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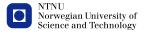
Q is positive definite if and only if the eigenvalues of A are all positive.



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- Q is positive definite if and only if the eigenvalues of A are all positive.
- Q is negative definite if and only if the eigenvalues of A are all negative.

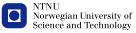


Quadratic forms and eigenvalues

Theorem 5

Let *A* be an $n \times n$ symmetric matrix, and let *Q* be the quadratic form $x \mapsto \mathbf{x}^t A \mathbf{x}$.

- Q is positive definite if and only if the eigenvalues of A are all positive.
- Q is negative definite if and only if the eigenvalues of A are all negative.
- Q is indefinite if and only if A has both positive and negative eigenvalues.



Problem 7 from June 2010

$$\operatorname{Let} A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

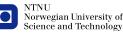
Find the eigenvalues and eigenvectors of A.

- Find a matrix *P* and a diagonal matrix *D* such that $A = PDP^{T}$.
- Solve the system of differential equations

$$y'_1 = 3y_1 + y_2 + y_3$$

 $y'_2 = y_1 + 2y_2$
 $y'_3 = y_1 + 2y_3$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$.



Solution



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Solution

$$det(A - \lambda I_3) = \begin{vmatrix} 3 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 0 \\ 1 & 0 & 2 - \lambda \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 2 - \lambda \\ 1 & 0 \end{vmatrix} + (2 - \lambda) \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix}$$
$$= -(2 - \lambda) + (2 - \lambda)((3 - \lambda)(2 - \lambda) - 1)$$
$$= (2 - \lambda)((3 - \lambda)(2 - \lambda) - 2)$$
$$= (2 - \lambda)(\lambda^2 - 5\lambda - 4)$$



Solution

$$\det(A - \lambda I_3) = \begin{vmatrix} 3 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 0 \\ 1 & 0 & 2 - \lambda \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 2 - \lambda \\ 1 & 0 \end{vmatrix} + (2 - \lambda) \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix}$$
$$= -(2 - \lambda) + (2 - \lambda)((3 - \lambda)(2 - \lambda) - 1)$$
$$= (2 - \lambda)((3 - \lambda)(2 - \lambda) - 2)$$
$$= (2 - \lambda)(\lambda^2 - 5\lambda - 4)$$
and $\lambda^2 - 5\lambda - 4 = 0$ if and only if
$$\lambda = \frac{5 \pm \sqrt{5^2 - 4(-4)}}{2} = \frac{5 \pm 3^2}{2} = \begin{cases} 4 \\ 1 \end{cases}$$
,
$$\bigcap_{\text{Norwegian University of Science and Technology} \end{cases}$$

λ

so the eigenvalues of A are 1,2 and 4.



so the eigenvalues of *A* are 1,2 and 4. To find the eigenvectors of *A* corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.



so the eigenvalues of *A* are 1,2 and 4. To find the eigenvectors of *A* corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.

$$A - I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$



so the eigenvalues of *A* are 1,2 and 4. To find the eigenvectors of *A* corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.

$$A - I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set of eigenvectors of *A* corresponding to 1 is $\left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.



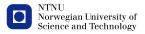
$$A - 2I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



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$$A - 2I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ so the set of eigenvectors of } A$$

corresponding to 2 is
$$\begin{cases} t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : t \neq 0 \end{cases}.$$

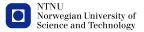


To find the eigenvectors of *A* corresponding to 4, we reduce $A - 4I_3$ to its reduced echelon form.



To find the eigenvectors of *A* corresponding to 4, we reduce $A - 4I_3$ to its reduced echelon form.

$$A - 4I_{3} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$



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$$A - 4I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set of eigenvectors of *A* corresponding to 4 $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

is
$$\left\{ \begin{array}{c} t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$$
.



To find a matrix *P* and a diagonal matrix *D* such that $A = PDP^{T}$ we will orthogonally diagonalize *A*.

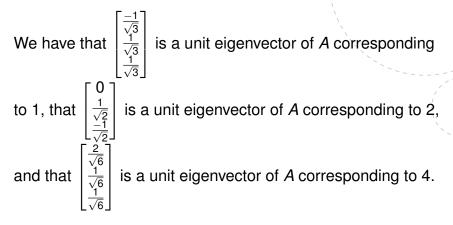


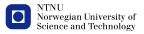
To find a matrix *P* and a diagonal matrix *D* such that $A = PDP^{T}$ we will orthogonally diagonalize *A*. For that we need an orthonormal basis for \mathbb{R}^{3} consisting of eigenvectors of *A*.



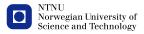
To find a matrix P and a diagonal matrix D such that $A = PDP^{T}$ we will orthogonally diagonalize A. For that we need an orthonormal basis for \mathbb{R}^{3} consisting of eigenvectors of A. Since A is symmetric, eigenvectors of A corresponding to different eigenvalues are orthogonal to each other, so we just have to find a unit vector in each of the 3 eigenspaces of A.







So if we let
$$P = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$
 and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,



So if we let
$$P = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$
 and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A = PDP^{-1} = PDP^{T}$.

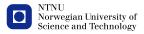


The coefficient matrix the system

$$y'_1 = 3y_1 + y_2 + y_3$$

 $y'_2 = y_1 + 2y_2$
 $y'_3 = y_1 + 2y_3$

is A,



so since the eigenvalues of *A* are 1,2 and 4, and $\begin{bmatrix} -1\\1\\1\end{bmatrix}$ is an eigenvector of *A* corresponding to 1, $\begin{bmatrix} 0\\1\\-1 \end{bmatrix}$ is an eigenvector of A corresponding to 2, and $\begin{bmatrix} 2\\1\\1 \end{bmatrix}$ is an eigenvector of A corresponding to 4,



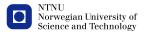
so since the eigenvalues of *A* are 1,2 and 4, and $\begin{bmatrix} -1\\1\\1 \end{bmatrix}$ is an eigenvector of A corresponding to 1, $\begin{bmatrix} 0\\1\\-1 \end{bmatrix}$ is an eigenvector of *A* corresponding to 2, and $\begin{bmatrix} 2\\1\\1 \end{bmatrix}$ is an eigenvector of *A* corresponding to 4, the general solution of the system is $\begin{vmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{vmatrix} = c_1 \begin{vmatrix} -1 \\ 1 \\ 1 \end{vmatrix} e^t + c_2 \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix} e^{2t} + c_3 \begin{vmatrix} 2 \\ 1 \\ 0 \\ Norwegian Universe} e^{4t}.$ Science and Technology

If $\begin{vmatrix} y_1(t) \\ y_2(t) \\ y_2(t) \end{vmatrix} = c_1 \begin{vmatrix} -1 \\ 1 \\ 1 \end{vmatrix} e^t + c_2 \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{vmatrix} e^{4t}$, then $\begin{vmatrix} y_{1}(0) \\ y_{2}(0) \\ y_{3}(0) \end{vmatrix} = c_{1} \begin{vmatrix} -1 \\ 1 \\ 1 \end{vmatrix} + c_{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{vmatrix} + c_{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{vmatrix}$ $= \begin{vmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_1 \end{bmatrix}$



so $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$ if and only if

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$



so $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$ if and only if

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We reduce the augmented matrix of the above system to its reduced echelon form.



$$\begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We see that the solution to the system

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

s. $c_1 = -1, c_2 = 2, c_3 = 1.$

So the solution to the system of differential equations

$$y'_1 = 3y_1 + y_2 + y_3$$

 $y'_2 = y_1 + 2y_2$
 $y'_3 = y_1 + 2y_3$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$, is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} 2e^{4t} + e^t \\ e^{4t} + 2e^{2t} - e^t \\ e^{4t} - 2e^{2t} - e^t \end{bmatrix}$$

