NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 26, April 18

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday’s lecture

NTNU
Norwegian University of
Science and Technology

Yesterday's lecture

Yesterday we introduced and studied

Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,

NTNU
Norwegian University of Science and Technology

Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,

Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,

NTNU
Norwegian University of
Science and Technology

Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- the *orthogonal complement* of a subspace,

Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- the *orthogonal complement* of a subspace,
- *orthogonal bases* and *orthonormal bases*,

NTNU
Norwegian University of
Science and Technology

Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- the *orthogonal complement* of a subspace,
- *orthogonal bases* and *orthonormal bases*,
- *orthogonal* matrices.

Today's lecture

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall introduce and study

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall introduce and study

- *orthogonal projections*,

NTNU
Norwegian University of Science and Technology

Today's lecture

Today we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

Orthogonal complements

Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$.

Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.

NTNU
Norwegian University of Science and Technology

Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.
$W^\perp$ is a subspace of $\mathbb{R}^n$.

NTNU
Norwegian University of Science and Technology

The orthogonal complements of Row(A) and Col(A)

Theorem 3

Let A be an $m \times n$ matrix. Then

1. $(\text{Row}(A))^{\perp} = \text{Nul}(A)$.
2. $(\text{Col}(A))^{\perp} = \text{Nul}(A^T)$.

NTNU
Norwegian University of Science and Technology

Orthogonal sets and bases

Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.

NTNU
Norwegian University of
Science and Technology

Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

NTNU
Norwegian University of Science and Technology

Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Theorem 5

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in W.

Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Theorem 5

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in W. Then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

NTNU
Norwegian University of Science and Technology

Orthonormal sets and bases

Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.

NTNU
Norwegian University of
Science and Technology

Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

NTNU
Norwegian University of
Science and Technology

Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.

NTNU
Norwegian University of Science and Technology

Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if
  $$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if
  $$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.
- If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and $\mathbf{y}$ is in W, then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \mathbf{y} \cdot \mathbf{v}_j$.

NTNU
Norwegian University of Science and Technology

Matrices with orthonormal columns

Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
2. $\|U\mathbf{x}\| = \|\mathbf{x}\|$.
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

NTNU
Norwegian University of Science and Technology

Orthogonal matrices

Orthogonal matrices

A square matrix with **orthonormal** columns is called an *orthogonal matrix*.

NTNU
Norwegian University of Science and Technology

Orthogonal matrices

A square matrix with **orthonormal** columns is called an *orthogonal matrix*.
A square matrix U is orthogonal if and only if U is invertible and $U^{-1} = U^T$.

NTNU
Norwegian University of
Science and Technology

Orthogonal projections

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.

NTNU
Norwegian University of
Science and Technology

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.

NTNU
Norwegian University of
Science and Technology

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L\, \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).

NTNU
Norwegian University of Science and Technology

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L\, \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).
- If $\mathbf{z} = \mathbf{y} - \text{proj}_L\, \mathbf{y}$, then $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y} = \text{proj}_L\, \mathbf{y} + \mathbf{z}$.

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L\, \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).
- If $\mathbf{z} = \mathbf{y} - \text{proj}_L\, \mathbf{y}$, then $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y} = \text{proj}_L\, \mathbf{y} + \mathbf{z}$.
- The vector $\mathbf{z} = \mathbf{y} - \text{proj}_L\, \mathbf{y}$ is called the component of $\mathbf{y}$ orthogonal to L (or to $\mathbf{u}$).

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).
- If $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$, then $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y} = \text{proj}_L \mathbf{y} + \mathbf{z}$.
- The vector $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$ is called the component of $\mathbf{y}$ orthogonal to L (or to $\mathbf{u}$).
- $\|z\|$ is called the *distance from* $\boldsymbol{y}$ *to* L.

NTNU
Norwegian University of
Science and Technology

Example

Let $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let us compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

NTNU
Norwegian University of
Science and Technology

Solution

NTNU
Norwegian University of
Science and Technology

Solution

The line through $\mathbf{u}$ and the origin is the line $L = \text{Span}\{\mathbf{u}\}$.

NTNU
Norwegian University of
Science and Technology

Solution

The line through $\mathbf{u}$ and the origin is the line $L = \text{Span}\{\mathbf{u}\}$.

$$\text{proj}_L\, \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$$

NTNU
Norwegian University of Science and Technology

Solution

The line through $\mathbf{u}$ and the origin is the line $L = \text{Span}\{\mathbf{u}\}$.

$$\text{proj}_L\,\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}7/2\\7/2\end{bmatrix}$$

so the distance from $\mathbf{y}$ to L is

$$\begin{aligned}\|\mathbf{y} - \text{proj}_L\,\mathbf{y}\| &= \left\|\begin{bmatrix}3/2\\-3/2\end{bmatrix}\right\| \\ &= \sqrt{(3/2)^2 + (-3/2)^2} = \sqrt{9/2} = \frac{3}{\sqrt{2}}.\end{aligned}$$

NTNU
Norwegian University of
Science and Technology

The orthogonal decomposition theorem

NTNU
Norwegian University of Science and Technology

The orthogonal decomposition theorem

Theorem 8

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

NTNU
Norwegian University of
Science and Technology

The orthogonal decomposition theorem

Theorem 8

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\mathbf{y}$ can be written uniquely in the form $\mathbf{y} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

NTNU
Norwegian University of Science and Technology

The orthogonal decomposition theorem

Theorem 8

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\mathbf{y}$ can be written uniquely in the form $\mathbf{y} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w}$ is in W and $\mathbf{z}$ is in $W^\perp$.
2. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W, then $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ and $\mathbf{z} = \mathbf{y} - \mathbf{w}$.

NTNU
Norwegian University of Science and Technology

The orthogonal decomposition theorem

Theorem 8

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\mathbf{y}$ can be written uniquely in the form $\mathbf{y} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w}$ is in W and $\mathbf{z}$ is in $W^\perp$.
2. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W, then $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ and $\mathbf{z} = \mathbf{y} - \mathbf{w}$.

The vector $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ is called the *orthogonal projection of* $\mathbf{y}$ *onto* W and is denoted by $\text{proj}_W\,\mathbf{y}$.

Proof of Theorem 8

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.

Proof of Theorem 8

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for W and let
$\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let
$\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then
$\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$ for each k, so $\mathbf{z}$ is in $W^{\perp}$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for W and let
$\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then
$\mathbf{z}\cdot\mathbf{u}_k = \mathbf{y}\cdot\mathbf{u}_k - \mathbf{y}\cdot\mathbf{u}_k = 0$ for each k, so $\mathbf{z}$ is in $W^\perp$.
Suppose $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$ where $\mathbf{w}'$ is in W and $\mathbf{z}'$ is in $W^\perp$.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then
$\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$ for each k, so $\mathbf{z}$ is in $W^\perp$.
Suppose $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$ where $\mathbf{w}'$ is in W and $\mathbf{z}'$ is in $W^\perp$. Then $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z}$ is both in W and in $W^\perp$,

NTNU
Norwegian University of Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let
$\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then
$\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$ for each k, so $\mathbf{z}$ is in $W^{\perp}$.
Suppose $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$ where $\mathbf{w}'$ is in W and $\mathbf{z}'$ is in $W^{\perp}$. Then
$\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z}$ is both in W and in $W^{\perp}$, so
$\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z} = \mathbf{0}$,

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then
$\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$ for each k, so $\mathbf{z}$ is in $W^\perp$.
Suppose $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$ where $\mathbf{w}'$ is in W and $\mathbf{z}'$ is in $W^\perp$. Then $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z}$ is both in W and in $W^\perp$, so $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z} = \mathbf{0}$, from which it follows that $\mathbf{w} = \mathbf{w}'$ and $\mathbf{z} = \mathbf{z}'$.

NTNU
Norwegian University of Science and Technology

The best approximation theorem

The best approximation theorem

Theorem 9

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be in $\mathbb{R}^n$, and let $\mathbf{w} = \text{proj}_W \mathbf{y}$.

The best approximation theorem

Theorem 9

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be in $\mathbb{R}^n$, and let $\mathbf{w} = \text{proj}_W \mathbf{y}$.
Then $\mathbf{w}$ is the closest point in W to $\mathbf{y}$ in the sense that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}$ in W distinct from $\mathbf{w}$.

Proof of Theorem 9

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 9

Let $\mathbf{x}$ be a vector in W distinct from $\mathbf{w}$.

Proof of Theorem 9

Let $\mathbf{x}$ be a vector in W distinct from $\mathbf{w}$. Then $\mathbf{w} - \mathbf{x}$ is in W and $\mathbf{y} - \mathbf{w}$ is in $W^{\perp}$,

Proof of Theorem 9

Let $\mathbf{x}$ be a vector in W distinct from $\mathbf{w}$. Then $\mathbf{w} - \mathbf{x}$ is in W and $\mathbf{y} - \mathbf{w}$ is in $W^{\perp}$, so $\mathbf{w} - \mathbf{x}$ and $\mathbf{y} - \mathbf{w}$ are orthogonal.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 9

Let $\mathbf{x}$ be a vector in W distinct from $\mathbf{w}$. Then $\mathbf{w} - \mathbf{x}$ is in W and $\mathbf{y} - \mathbf{w}$ is in $W^{\perp}$, so $\mathbf{w} - \mathbf{x}$ and $\mathbf{y} - \mathbf{w}$ are orthogonal. It follows that

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|(\mathbf{y} - \mathbf{w}) + (\mathbf{w} - \mathbf{x})\|^2 = \|\mathbf{y} - \mathbf{w}\|^2 + \|\mathbf{w} - \mathbf{x}\|^2.$$

Proof of Theorem 9

Let $\mathbf{x}$ be a vector in W distinct from $\mathbf{w}$. Then $\mathbf{w} - \mathbf{x}$ is in W and $\mathbf{y} - \mathbf{w}$ is in $W^\perp$, so $\mathbf{w} - \mathbf{x}$ and $\mathbf{y} - \mathbf{w}$ are orthogonal. It follows that

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|(\mathbf{y} - \mathbf{w}) + (\mathbf{w} - \mathbf{x})\|^2 = \|\mathbf{y} - \mathbf{w}\|^2 + \|\mathbf{w} - \mathbf{x}\|^2.$$

Since $\mathbf{x} \neq \mathbf{w}$, it follows that $\|\mathbf{w} - \mathbf{x}\| > 0$, and thus that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$.

Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us find the closest point in W to $\mathbf{y}$.

NTNU
Norwegian University of
Science and Technology

Solution

Solution

The closest point in W to $\mathbf{y}$ is $\text{proj}_W\, \mathbf{y}$.

Solution

The closest point in W to $\mathbf{y}$ is $\text{proj}_W \mathbf{y}$. We have that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W.

NTNU
Norwegian University of Science and Technology

Solution

The closest point in W to $\mathbf{y}$ is $\text{proj}_W\,\mathbf{y}$. We have that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W. It follows that

$$\begin{aligned}
\text{proj}_W\,\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 \\
&= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.
\end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Orthogonal projections onto orthonormal bases

NTNU
Norwegian University of Science and Technology

Orthogonal projections onto orthonormal bases

Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

Orthogonal projections onto orthonormal bases

Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$.

Orthogonal projections onto orthonormal bases

Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$.
2. If $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$, then $\text{proj}_W \mathbf{y} = UU^T\mathbf{y}$.

Proof of Theorem 10

Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W.

Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W. Then $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for each k,

Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W. Then $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for each k, so
$\text{proj}_W \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p$
for each $\mathbf{y}$ in $\mathbb{R}^n$.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W. Then $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for each k, so
$\text{proj}_W \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p$
for each $\mathbf{y}$ in $\mathbb{R}^n$.
Let $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$.

Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W. Then $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for each k, so
$\text{proj}_W \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p$
for each $\mathbf{y}$ in $\mathbb{R}^n$.
Let $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$. Then

$$
\begin{aligned}
UU^T\mathbf{y} = [\mathbf{u}_1 \ldots \mathbf{u}_p][\mathbf{u}_1 \ldots \mathbf{u}_p]^T\mathbf{y} &= [\mathbf{u}_1 \ldots \mathbf{u}_p]\begin{bmatrix}\mathbf{u}_1\cdot\mathbf{y}\\ \vdots \\ \mathbf{u}_p\cdot\mathbf{y}\end{bmatrix} \\
&= (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p = \text{proj}_W \mathbf{y}
\end{aligned}
$$

for each $\mathbf{y}$ in $\mathbb{R}^n$.

NTNU
Norwegian University of
Science and Technology

Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us use Theorem 10 to find the closest point in W to $\mathbf{y}$.

NTNU
Norwegian University of
Science and Technology

Solution

NTNU
Norwegian University of Science and Technology

Solution

$\|\mathbf{u}_1\|^2 = \mathbf{u}_1 \cdot \mathbf{u}_1 = 30$ and $\|\mathbf{u}_2\|^2 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 6$,

NTNU
Norwegian University of
Science and Technology

Solution

$\|\mathbf{u}_1\|^2 = \mathbf{u}_1 \cdot \mathbf{u}_1 = 30$ and $\|\mathbf{u}_2\|^2 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 6$, so

$\left\{ \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for W.

NTNU
Norwegian University of
Science and Technology

Solution

$\|\mathbf{u}_1\|^2 = \mathbf{u}_1 \cdot \mathbf{u}_1 = 30$ and $\|\mathbf{u}_2\|^2 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 6$, so

$$\left\{ \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$ is an orthonormal basis for W.

Let $U = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}$.

Solution

Then

$$UU^T = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution

Then

$$UU^T = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$

so $\text{proj}_W \mathbf{y} = UU^T\mathbf{y} = \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$.

NTNU
Norwegian University of
Science and Technology

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

NTNU
Norwegian University of Science and Technology

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

NTNU
Norwegian University of Science and Technology

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$.

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
3. If appropriate, scale $\mathbf{v}_2$ to simplify later calculations.

NTNU
Norwegian University of Science and Technology

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}\mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$ and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
3. If appropriate, scale $\mathbf{v}_2$ to simplify later calculations.
4. Let $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2}\mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$, let $W_3 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and scale $\mathbf{v}_3$ to simplify later calculations (if appropriate).

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

5. Continue like this and produce vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ where, for $1 < k \leq p$, $\mathbf{v}_k$ is an appropriate multiple of $\mathbf{x}_k - \text{proj}_{\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k - \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\mathbf{v}_{k-1}$.

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

5. Continue like this and produce vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ where, for $1 < k \leq p$, $\mathbf{v}_k$ is an appropriate multiple of $\mathbf{x}_k - \text{proj}_{\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k - \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\mathbf{v}_{k-1}$.
6. Then $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is an orthogonal basis for W.

NTNU
Norwegian University of Science and Technology

The Gram-Schmidt process

The Gram-Schmidt process

Theorem 11

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a linearly independent subset of $\mathbb{R}^n$. Let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_k - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

NTNU
Norwegian University of Science and Technology

The Gram-Schmidt process

Theorem 11

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a linearly independent subset of $\mathbb{R}^n$. Let

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_k - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}
$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set, and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $1 \leq k \leq p$.

Proof of Theorem 11

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.
Suppose that $1 \leq k < p$ and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W_k.

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.
Suppose that $1 \leq k < p$ and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W_k. Then $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$ is orthogonal to W_k and in W_{k+1},

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \le k \le p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.

Suppose that $1 \le k < p$ and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W_k. Then $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$ is orthogonal to W_k and in W_{k+1}, so $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}$ is an orthogonal set in W_{k+1}.

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \le k \le p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.

Suppose that $1 \le k < p$ and that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W_k. Then $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$ is orthogonal to W_k and in W_{k+1}, so $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ is an orthogonal set in W_{k+1}. Since $\dim(W_{k+1}) = k + 1$, it follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ is an orthogonal basis for W_{k+1}.

Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \le k \le p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.

Suppose that $1 \le k < p$ and that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W_k. Then $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$ is orthogonal to W_k and in W_{k+1}, so $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ is an orthogonal set in W_{k+1}. Since $\dim(W_{k+1}) = k + 1$, it follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ is an orthogonal basis for W_{k+1}.

It follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W_k for all $1 \le k \le p$.

NTNU
Norwegian University of
Science and Technology

Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and
$W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.
Let us find an orthogonal basis for W.

Solution

NTNU
Norwegian University of
Science and Technology

Solution

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W.

NTNU
Norwegian University of Science and Technology

Solution

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$.

NTNU
Norwegian University of
Science and Technology

Solution

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$.
Let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \tfrac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \tfrac{1}{4} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

NTNU
Norwegian University of
Science and Technology

Solution

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$.
Let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}\mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2' = 4\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

NTNU
Norwegian University of
Science and Technology

Solution

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$.
Let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}\mathbf{x}_2 = \mathbf{x}_2 - \tfrac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \tfrac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \tfrac{1}{4}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},$$

$$\mathbf{v}_2' = 4\mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \text{ and}$$

$W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2'\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

NTNU
Norwegian University of
Science and Technology

Solution

Let

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{x}_3 - \mathrm{proj}_{W_2}\,\mathbf{x}_3 \\ &= x_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2' \\ &= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix}0\\-2\\1\\1\end{bmatrix}\end{aligned}$$

Solution (cont.)

and $\mathbf{v}_3' = 3\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$.

Solution (cont.)

and $\mathbf{v}_3' = 3\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$ is an orthogonal basis

for W.

NTNU
Norwegian University of Science and Technology

The QR factorization

The QR factorization

Theorem 12

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where

NTNU
Norwegian University of Science and Technology

The QR factorization

Theorem 12

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where

- Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{Col}(A)$,

NTNU
Norwegian University of
Science and Technology

The QR factorization

Theorem 12

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where

- Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{Col}(A)$,
- R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 12

Proof of Theorem 12

Let $A = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set such that $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 12

Let $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$ and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthogonal set such that $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$.

Then we have for each $1 \leq k \leq p$ that there exist scalars $r_{1k}, r_{2k}, \dots, r_{kk}$ such that $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \dots r_{kk}\mathbf{v}_k$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 12

Let $A = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set such that $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$.
Then we have for each $1 \leq k \leq p$ that there exist scalars $r_{1k}, r_{2k}, \ldots, r_{kk}$ such that $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \ldots r_{kk}\mathbf{v}_k$.
We must have that $r_{kk} \neq 0$ because otherwise $\mathbf{x}_k$ would be in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k_1}\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{k_1}\}$ which would contradict the assumption that $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is linearly independent.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 12

Let $A = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set such that $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$.

Then we have for each $1 \leq k \leq p$ that there exist scalars $r_{1k}, r_{2k}, \ldots, r_{kk}$ such that $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \ldots r_{kk}\mathbf{v}_k$.

We must have that $r_{kk} \neq 0$ because otherwise $\mathbf{x}_k$ would be in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k_1}\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{k_1}\}$ which would contradict the assumption that $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is linearly independent.

We may assume that $r_{kk} > 0$, because if $r_{kk} < 0$, then we can replace $\mathbf{v}_k$ by $-\mathbf{v}_k$ and r_{kk} by $-r_{kk}$, and then $r_{kk} > 0$.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 12 (cont.)

Let $Q = [\mathbf{v}_1 \ldots \mathbf{v}_n]$ and $R = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ 0 & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & r_{nn} \end{bmatrix}$.

NTNU
Norwegian University of Science and Technology

Proof of Theorem 12 (cont.)

Let $Q = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$.

Then $QR = [\mathbf{x}_1 \dots \mathbf{x}_n] = A$.

NTNU
Norwegian University of Science and Technology

Example

Let us find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

NTNU
Norwegian University of
Science and Technology

Solution

Solution

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$\mathbf{v}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$.

NTNU
Norwegian University of Science and Technology

Solution (cont.)

Then $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set such that

$$\begin{aligned}
\mathbf{x}_1 &= 2\mathbf{v}_1 \\
\mathbf{x}_2 &= \frac{3}{2}\mathbf{v}_1 + \frac{\sqrt{3}}{2}\mathbf{v}_2 \\
\mathbf{x}_3 &= \mathbf{v}_1 + \frac{1}{\sqrt{3}}\mathbf{v}_2 + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{v}_3
\end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

So if we let $Q = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and

$R = \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$, then $QR = A$ is a QR factorization of A.

NTNU
Norwegian University of
Science and Technology

Problem 6 from June 2010

Let A be the following matrix; find a basis for each of the spaces $\mathrm{Nul}(A)$, $\mathrm{Col}(A)$, $(\mathrm{Col}(A))^{\perp}$, and $\mathrm{Row}(A)$.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix}$$

Find the orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to $\mathrm{Col}(A)$.

NTNU
Norwegian University of Science and Technology

Solution

Solution

We start by reducing A to its reduced echelon form.

NTNU
Norwegian University of Science and Technology

Solution

We start by reducing A to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 2 & -6 & -8 & -8 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution

We see that $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for

$\mathrm{Nul}(A)$,

Solution

We see that $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul(A), that $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for Col(A),

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

and that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ -4 \\ -4 \end{bmatrix} \right\}$ is a basis for Row(A).

NTNU
Norwegian University of Science and Technology

Solution (cont.)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ is in $(\mathrm{Col}(A))^\perp$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = x_1 + 3x_2 + 4x_3 = 0 \text{ and}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = x_2 + 2x_3 = 0.$$

Solution (cont.)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $(\mathrm{Col}(A))^\perp$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = x_1 + 3x_2 + 4x_3 = 0 \text{ and}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = x_2 + 2x_3 = 0.$$

We reduce the coefficient matrix of the system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

to its reduced echelon form.

Solution (cont.)

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

We see that $(\mathrm{Col}(A))^\perp = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$, so $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $(\mathrm{Col}(A))^\perp$.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

The orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to $(\mathrm{Col}(A))^\perp$ is

$$\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Solution (cont.)

The orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to $(\mathrm{Col}(A))^{\perp}$ is

$$\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \tfrac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$ so the orthogonal projection of

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to $\mathrm{Col}(A)$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \tfrac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \tfrac{1}{9} \begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix}$.

NTNU
Norwegian University of
Science and Technology

Problem 5 from June 2011

Let V be the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

and let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}.$$

Find the nearest point in V to $\mathbf{b}$ (that is, the orthogonal projection of $\mathbf{b}$ on to V).

NTNU
Norwegian University of
Science and Technology

Solution

Solution

We start by reducing the matrix to an echelon form.

Solution

We start by reducing the matrix to an echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution

We start by reducing the matrix to an echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for V.

NTNU
Norwegian University of
Science and Technology

Solution

We start by reducing the matrix to an echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for V. We then find an orthogonal basis for V by using the Gram-Schmidt process on $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$ is an orthogonal basis for V.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

We then have that

$$\operatorname{proj}_V \mathbf{b} = \frac{\begin{bmatrix}1\\7\\3\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-1\end{bmatrix}}{\begin{bmatrix}1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-1\end{bmatrix}} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{\begin{bmatrix}1\\7\\3\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\0\end{bmatrix}}{\begin{bmatrix}2\\-1\\0\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\0\end{bmatrix}} \begin{bmatrix}2\\-1\\0\end{bmatrix}$$

$$= \frac{12}{6} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{-5}{5} \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\5\\-2\end{bmatrix}.$$

NTNU
Norwegian University of
Science and Technology

Problem 5 from December 2010

Let $V \subseteq \mathbb{R}^4$ be the solution space of the linear system

$$\begin{aligned} x + y - z + w &= 0 \\ x + 2y - 2z + w &= 0 \end{aligned}$$

1. Find an orthogonal basis for V.
2. Find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ on to V.
3. Find an orthogonal basis for $\mathbb{R}^4$ in which the first two first basis vectors are the once we found in (1).

NTNU
Norwegian University of
Science and Technology

Solution

Solution

We start by reducing the coefficient matrix of the system to its reduced echelon form.

NTNU
Norwegian University of Science and Technology

Solution

We start by reducing the coefficient matrix of the system to its reduced echelon form.

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

NTNU
Norwegian University of Science and Technology

Solution

We start by reducing the coefficient matrix of the system to its reduced echelon form.

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

We see that $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for V and that it is orthogonal.

NTNU
Norwegian University of Science and Technology

Solution (cont.)

The orthogonal projection of **b** on to V is

$$\frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}+\frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

Let A be the coefficient matrix of the system.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

Let A be the coefficient matrix of the system. Then

$$V^{\perp} = (\mathrm{Nul}(A))^{\perp} = \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\},$$

Solution (cont.)

Let A be the coefficient matrix of the system. Then

$$V^\perp = (\text{Nul}(A))^\perp = \text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}, \text{ so}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ is an orthogonal basis for } \mathbb{R}^4.$$

Plan for next week

Plan for next week

Wednesday we shall look at

- *least-squares problems*,
- applications to linear models.

Sections 6.5–6.6 in “Linear Algebras and Its Applications” (pages 360–375).

Plan for next week

Wednesday we shall look at

- *least-squares problems*,
- applications to linear models.

Sections 6.5–6.6 in “Linear Algebras and Its Applications” (pages 360–375).

Thursday we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

Sections 7.1–7.2 in “Linear Algebras and Its Applications” (pages 393–407).