

#### TMA4115 - Calculus 3 Lecture 26, April 18

Toke Meier Carlsen Norwegian University of Science and Technology Spring 2013



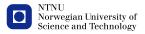
TMA4115 - Calculus 3, Lecture 26, page 2

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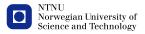


Yesterday we introduced and studied

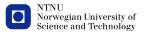
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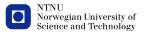
- the inner product,
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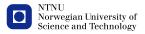
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- orthogonal matrices.





TMA4115 - Calculus 3, Lecture 26, page 3

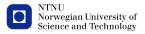
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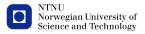
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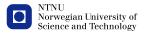
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- orthogonal projections,
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TMA4115 - Calculus 3, Lecture 26, page 4

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# The orthogonal complements of Row(A) and Col(A)

#### Theorem 3

Let *A* be an  $m \times n$  matrix. Then

$$(\operatorname{Row}(A))^{\perp} = \operatorname{Nul}(A)$$

$$(\operatorname{Col}(A))^{\perp} = \operatorname{Nul}(A^{T}).$$





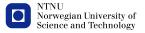
TMA4115 - Calculus 3, Lecture 26, page 6

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A set of vectors {**v**<sub>1</sub>,..., **v**<sub>p</sub>} in ℝ<sup>n</sup> is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if **v**<sub>i</sub> ⋅ **v**<sub>j</sub> = 0 whenever i ≠ j.



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#### Theorem 5

Let  $\mathcal{B} = {\mathbf{v}_1, \dots, \mathbf{v}_p}$  be an orthogonal basis for a subspace W of  $\mathbb{R}^n$  and let  $\mathbf{y}$  be in W.



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#### Theorem 5

Let  $\mathcal{B} = {\mathbf{v}_1, \dots, \mathbf{v}_p}$  be an orthogonal basis for a subspace W of  $\mathbb{R}^n$  and let  $\mathbf{y}$  be in W. Then the coordinates  $c_1, \dots, c_p$  of  $\mathbf{y}$  relative to  $\mathcal{B}$  is given by  $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$ .



TMA4115 - Calculus 3, Lecture 26, page 7

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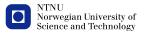
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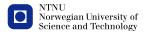
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- If B = {v<sub>1</sub>,..., v<sub>p</sub>} be an orthonormal basis for a subspace W of ℝ<sup>n</sup> and y is in W, then the coordinates c<sub>1</sub>,..., c<sub>p</sub> of y relative to B is given by c<sub>j</sub> = y ⋅ v<sub>j</sub>.



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## Matrices with orthonormal columns



TMA4115 - Calculus 3, Lecture 26, page 8

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## Matrices with orthonormal columns

#### Theorem 6

An  $m \times n$  matrix U has orthonormal columns if and only if  $U^T U = I_n$ .



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#### Theorem 7

Let *U* be an  $m \times n$  matrix with orthonormal columns and let **x** and **y** be in  $\mathbb{R}^n$ . Then

$$\bigcirc (U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}.$$

$$\| U \mathbf{x} \| = \| \mathbf{x} \|.$$

**3**  $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$  if and only if  $\mathbf{x} \cdot \mathbf{y} = 0$ .



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#### **Orthogonal matrices**



TMA4115 - Calculus 3, Lecture 26, page 9

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## **Orthogonal matrices**

A square matrix with **orthonormal** columns is called an *orthogonal matrix*.



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A square matrix U is orthogonal if and only if U is invertible and  $U^{-1} = U^{T}$ .





TMA4115 - Calculus 3, Lecture 26, page 10

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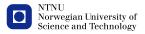
• Let **u** and **y** be vectors in  $\mathbb{R}^n$  and assume that  $\mathbf{u} \neq \mathbf{0}$ .



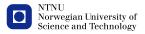
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- Let  $L = \text{Span}\{\mathbf{u}\}$ .



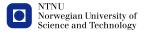
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- The vector  $\text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$  is called the *orthogonal* projection of  $\mathbf{y}$  onto L (or onto  $\mathbf{u}$ ).



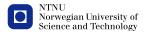
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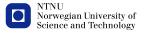


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- ||z|| is called the *distance from* **y** to *L*.



#### Example

Let  $\mathbf{y} = \begin{bmatrix} 5\\2 \end{bmatrix}$  and  $\mathbf{u} = \begin{bmatrix} 1\\1 \end{bmatrix}$ . Let us compute the distance from  $\mathbf{y}$  to the line through  $\mathbf{u}$  and the origin.





TMA4115 - Calculus 3, Lecture 26, page 12

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The line through **u** and the origin is the line  $L = \text{Span}\{\mathbf{u}\}$ .



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$$\operatorname{proj}_{L} \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{7}{2} \begin{bmatrix} 1\\1 \end{bmatrix} = \begin{bmatrix} 7/2\\7/2 \end{bmatrix}$$

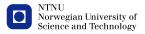


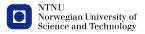
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so the distance from **y** to *L* is

$$\|\mathbf{y} - \operatorname{proj}_{L} \mathbf{y}\| = \left\| \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} \right\|$$
$$= \sqrt{(3/2)^{2} + (-3/2)^{2}} = \sqrt{9/2} = \frac{3}{\sqrt{2}}.$$





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Theorem 8

Let *W* be a subspace of  $\mathbb{R}^n$  and let **y** be in  $\mathbb{R}^n$ .



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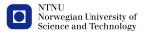
• Then **y** can be written uniquely in the form  $\mathbf{y} = \mathbf{w} + \mathbf{z}$ where **w** is in *W* and **z** is in  $W^{\perp}$ .



#### Theorem 8

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- Then y can be written uniquely in the form y = w + zwhere w is in W and z is in  $W^{\perp}$ .
- If  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is an orthogonal basis for W, then  $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$  and  $\mathbf{z} = \mathbf{y} - \mathbf{w}$ .

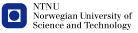


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The vector  $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$  is called the *orthogonal* projection of  $\mathbf{y}$  onto W and is denoted by  $\operatorname{proj}_W \mathbf{y}$ .





TMA4115 - Calculus 3, Lecture 26, page 14

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Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_{\rho}\}$  be an orthogonal basis for W and let  $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_{\rho}}{\mathbf{u}_{\rho} \cdot \mathbf{u}_{\rho}} \mathbf{u}_{\rho}.$ 



Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  be an orthogonal basis for W and let  $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$ . Then  $\mathbf{w}$  is in W.



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Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthogonal basis for W and let  $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$ . Then  $\mathbf{w}$  is in W. Let  $\mathbf{z} = \mathbf{y} - \mathbf{w}$ . Then  $\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$  for each k, so  $\mathbf{z}$  is in  $W^{\perp}$ . Suppose  $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$  where  $\mathbf{w}'$  is in W and  $\mathbf{z}'$  is in  $W^{\perp}$ . Then  $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z}$  is both in W and in  $W^{\perp}$ ,



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#### The best approximation theorem



TMA4115 - Calculus 3, Lecture 26, page 15

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### The best approximation theorem

#### Theorem 9

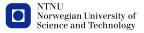
Let W be a subspace of  $\mathbb{R}^n$ , let  $\mathbf{y}$  be in  $\mathbb{R}^n$ , and let  $\mathbf{w} = \text{proj}_W \mathbf{y}$ .



## The best approximation theorem

#### Theorem 9

Let *W* be a subspace of  $\mathbb{R}^n$ , let **y** be in  $\mathbb{R}^n$ , and let  $\mathbf{w} = \operatorname{proj}_W \mathbf{y}$ . Then **w** is the closest point in *W* to **y** in the sense that  $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$  for all **x** in *W* distinct from **w**.





TMA4115 - Calculus 3, Lecture 26, page 16

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Let **x** be a vector in *W* distinct from **w**.



Let **x** be a vector in *W* distinct from **w**. Then  $\mathbf{w} - \mathbf{x}$  is in *W* and  $\mathbf{y} - \mathbf{w}$  is in  $W^{\perp}$ ,

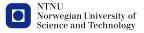


Let **x** be a vector in *W* distinct from **w**. Then  $\mathbf{w} - \mathbf{x}$  is in *W* and  $\mathbf{y} - \mathbf{w}$  is in  $W^{\perp}$ , so  $\mathbf{w} - \mathbf{x}$  and  $\mathbf{y} - \mathbf{w}$  are orthogonal.



Let **x** be a vector in *W* distinct from **w**. Then  $\dot{\mathbf{w}} - \mathbf{x}$  is in *W* and  $\mathbf{y} - \mathbf{w}$  is in  $W^{\perp}$ , so  $\mathbf{w} - \mathbf{x}$  and  $\mathbf{y} - \mathbf{w}$  are orthogonal. It follows that

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|(\mathbf{y} - \mathbf{w}) + (\mathbf{w} - \mathbf{x})\|^2 = \|\mathbf{y} - \mathbf{w}\|^2 + \|\mathbf{w} - \mathbf{x}\|^2$$



Let **x** be a vector in *W* distinct from **w**. Then  $\mathbf{w} - \mathbf{x}$  is in *W* and  $\mathbf{y} - \mathbf{w}$  is in  $W^{\perp}$ , so  $\mathbf{w} - \mathbf{x}$  and  $\mathbf{y} - \mathbf{w}$  are orthogonal. It follows that

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Since  $\mathbf{x} \neq \mathbf{w}$ , it follows that  $\|\mathbf{w} - \mathbf{x}\| > 0$ , and thus that  $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$ .



#### Example

## Let $\mathbf{u}_1 = \begin{bmatrix} 2\\5\\-1 \end{bmatrix}$ , $\mathbf{u}_2 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$ , $\mathbf{y} = \begin{bmatrix} 1\\2\\3 \end{bmatrix}$ , and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ . Let us find the closest point in W to $\mathbf{y}$ .





TMA4115 - Calculus 3, Lecture 26, page 18

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The closest point in W to **y** is  $proj_W$  **y**.



TMA4115 - Calculus 3, Lecture 26, page 18

The closest point in W to  $\mathbf{y}$  is  $\text{proj}_W \mathbf{y}$ . We have that  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ , so  $\{\mathbf{u}_1, \mathbf{u}_2\}$  is an orthogonal basis for W.

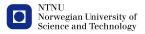


The closest point in W to **y** is  $\text{proj}_W$  **y**. We have that  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ , so  $\{\mathbf{u}_1, \mathbf{u}_2\}$  is an orthogonal basis for W. It follows that

р

$$\operatorname{roj}_{W} \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_{1}}{\mathbf{u}_{1} \cdot \mathbf{u}_{1}} \mathbf{u}_{1} + \frac{\mathbf{y} \cdot \mathbf{u}_{2}}{\mathbf{u}_{2} \cdot \mathbf{u}_{2}} \mathbf{u}_{2}$$
$$= \frac{9}{30} \begin{bmatrix} 2\\5\\-1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2\\1\\1 \end{bmatrix}$$
$$= \begin{bmatrix} -2/5\\2\\1/5 \end{bmatrix}.$$

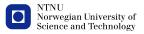




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#### Theorem 10

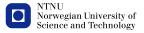
Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for a subspace W of  $\mathbb{R}^n$  and let  $\mathbf{y}$  be in  $\mathbb{R}^n$ .



#### Theorem 10

Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for a subspace W of  $\mathbb{R}^n$  and let  $\mathbf{y}$  be in  $\mathbb{R}^n$ .

**)** Then 
$$\operatorname{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$
.



#### Theorem 10

Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for a subspace W of  $\mathbb{R}^n$  and let  $\mathbf{y}$  be in  $\mathbb{R}^n$ .

**1** Then 
$$\operatorname{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$
.

**2** If 
$$U = [\mathbf{u}_1 \dots \mathbf{u}_{\rho}]$$
, then proj<sub>W</sub>  $\mathbf{y} = UU^T \mathbf{y}$ .





TMA4115 - Calculus 3, Lecture 26, page 20

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Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for W.



TMA4115 - Calculus 3, Lecture 26, page 20

Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for W. Then  $\mathbf{u}_k \cdot \mathbf{u}_k = 1$  for each k,



Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for W. Then  $\mathbf{u}_k \cdot \mathbf{u}_k = 1$  for each k, so proj<sub>W</sub>  $\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p = (\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p) \mathbf{u}_p$ for each  $\mathbf{y}$  in  $\mathbb{R}^n$ .



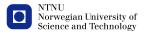
Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for W. Then  $\mathbf{u}_k \cdot \mathbf{u}_k = 1$  for each k, so proj<sub>W</sub>  $\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$ for each  $\mathbf{y}$  in  $\mathbb{R}^n$ . Let  $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$ .



Let  $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$  be an orthonormal basis for W. Then  $\mathbf{u}_k \cdot \mathbf{u}_k = 1$  for each k, so proj<sub>W</sub>  $\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$ for each  $\mathbf{y}$  in  $\mathbb{R}^n$ . Let  $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$ . Then

$$UU^{\mathsf{T}}\mathbf{y} = [\mathbf{u}_{1} \dots \mathbf{u}_{\rho}][\mathbf{u}_{1} \dots \mathbf{u}_{\rho}]^{\mathsf{T}}\mathbf{y} = [\mathbf{u}_{1} \dots \mathbf{u}_{\rho}]\begin{bmatrix}\mathbf{u}_{1} \cdot \mathbf{y}\\ \vdots\\ \mathbf{u}_{\rho} \cdot \mathbf{y}\end{bmatrix}$$
$$= (\mathbf{y} \cdot \mathbf{u}_{1})\mathbf{u}_{1} + \dots + (\mathbf{y} \cdot \mathbf{u}_{\rho})\mathbf{u}_{\rho} = \operatorname{proj}_{W}\mathbf{y}$$

for each **y** in  $\mathbb{R}^n$ .



#### Example

### Let $\mathbf{u}_1 = \begin{bmatrix} 2\\5\\-1 \end{bmatrix}$ , $\mathbf{u}_2 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$ , $\mathbf{y} = \begin{bmatrix} 1\\2\\3 \end{bmatrix}$ , and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ . Let us use Theorem 10 to find the closest point in W to $\mathbf{y}$ .





TMA4115 - Calculus 3, Lecture 26, page 22

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$$\|\mathbf{u}_1\|^2 = \mathbf{u}_1 \cdot \mathbf{u}_1 = 30 \text{ and } \|\mathbf{u}_2\|^2 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 6,$$



TMA4115 - Calculus 3, Lecture 26, page 22

$$\begin{aligned} \|\mathbf{u}_1\|^2 &= \mathbf{u}_1 \cdot \mathbf{u}_1 = 30 \text{ and } \|\mathbf{u}_2\|^2 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 6, \text{ so} \\ \left\{ \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\} \text{ is an orthonormal basis for } W. \end{aligned}$$

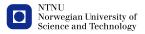


$$\begin{aligned} \|\mathbf{u}_{1}\|^{2} &= \mathbf{u}_{1} \cdot \mathbf{u}_{1} = 30 \text{ and } \|\mathbf{u}_{2}\|^{2} = \mathbf{u}_{2} \cdot \mathbf{u}_{2} = 6, \text{ so} \\ \left\{ \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\} \text{ is an orthonormal basis for } W. \\ \text{Let } U &= \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}. \end{aligned}$$



Then

$$UU^{T} = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$
$$= \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$



Then

$$UU^{T} = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$
$$= \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$
so proj<sub>W</sub>  $\mathbf{y} = UU^{T} \mathbf{y} = \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$ 

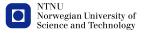
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Let  $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$  be a basis for a subspace W of  $\mathbb{R}^n$ .



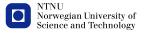
Let  $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$  be a basis for a subspace W of  $\mathbb{R}^n$ . **1** Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{v}_1\}$ .



Let  $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$  be a basis for a subspace W of  $\mathbb{R}^n$ .

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 and  $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$ 

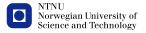


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If appropriate, scale  $\mathbf{v}_2$  to simplify later calculations.



Let  $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$  be a basis for a subspace W of  $\mathbb{R}^n$ .

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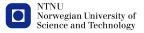
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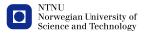
Let 
$$\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$$
, let  $W_3 = \operatorname{Span}{\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}}$ , and scale  $\mathbf{v}_3$  to simplify later calculations (if appropriate).



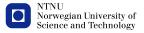
• Continue like this and produce vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{\bar{p}}$  where, for  $1 < k \le p$ ,  $\mathbf{v}_k$  is an appropriate multiple of  $\mathbf{x}_k - \operatorname{proj}_{\operatorname{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k - \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \dots - \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}} \mathbf{v}_{k-1}$ .



- Continue like this and produce vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{\bar{p}}$  where, for  $1 < k \le p$ ,  $\mathbf{v}_k$  is an appropriate multiple of  $\mathbf{x}_k \operatorname{proj}_{\operatorname{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \dots \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}} \mathbf{v}_{k-1}$ .
- Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is an orthogonal basis for W.



#### The Gram-Schmidt process



TMA4115 - Calculus 3, Lecture 26, page 26

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### **The Gram-Schmidt process**

#### Theorem 11

Let  $\{\mathbf{x}_1, \dots, \mathbf{x}_{\rho}\}$  be a linearly independent subset of  $\mathbb{R}^n$ . Let

$$\mathbf{v}_{1} = \mathbf{x}_{1}$$

$$\mathbf{v}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}}$$

$$\vdots$$

$$\mathbf{v}_{p} = \mathbf{x}_{k} - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \dots - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1}$$



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### The Gram-Schmidt process

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Let  $\{\mathbf{x}_1, \dots, \mathbf{x}_{\rho}\}$  be a linearly independent subset of  $\mathbb{R}^n$ . Let

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$$\vdots$$

$$\mathbf{v}_{p} = \mathbf{x}_{k} - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \dots - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1}$$

Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is an orthogonal set, and Span $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} =$ Span $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for  $1 \le k \le p$ .



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TMA4115 - Calculus 3, Lecture 26, page 27

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Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ .



Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ . Then  $\{\mathbf{v}_1\}$  is an orthogonal basis for  $W_1$ .



Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ . Then  $\{\mathbf{v}_1\}$  is an orthogonal basis for  $W_1$ . Suppose that  $1 \le k < p$  and that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis for  $W_k$ .



Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ . Then  $\{\mathbf{v}_1\}$  is an orthogonal basis for  $W_1$ . Suppose that  $1 \le k < p$  and that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis for  $W_k$ . Then  $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$  is orthogonal to  $W_k$  and in  $W_{k+1}$ ,



Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ . Then  $\{\mathbf{v}_1\}$  is an orthogonal basis for  $W_1$ . Suppose that  $1 \le k < p$  and that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis for  $W_k$ . Then  $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$  is orthogonal to  $W_k$  and in  $W_{k+1}$ , so  $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$  is an orthogonal set in  $W_{k+1}$ .



Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ . Then  $\{\mathbf{v}_1\}$  is an orthogonal basis for  $W_1$ . Suppose that  $1 \le k < p$  and that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis for  $W_k$ . Then  $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$  is orthogonal to  $W_k$  and in  $W_{k+1}$ , so  $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$  is an orthogonal set in  $W_{k+1}$ . Since dim $(W_{k+1}) = k + 1$ , it follows that  $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$  is an orthogonal basis for  $W_{k+1}$ .



Let  $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ . Then  $\{\mathbf{v}_1\}$  is an orthogonal basis for  $W_1$ . Suppose that  $1 \le k < p$  and that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis for  $W_k$ . Then  $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$  is orthogonal to  $W_k$  and in  $W_{k+1}$ , so  $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$  is an orthogonal set in  $W_{k+1}$ . Since dim $(W_{k+1}) = k + 1$ , it follows that  $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$  is an orthogonal basis for  $W_{k+1}$ . It follows that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis for  $W_k$  for all  $1 \le k \le p$ .



### Example

Let 
$$\mathbf{x}_1 = \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix}$$
,  $\mathbf{x}_2 = \begin{bmatrix} 0\\1\\1\\1 \end{bmatrix}$ ,  $\mathbf{x}_3 = \begin{bmatrix} 0\\0\\1\\1 \end{bmatrix}$ , and  
 $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ .  
Let us find an orthogonal basis for  $W$ .

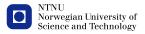




TMA4115 - Calculus 3, Lecture 26, page 29

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 $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent and therefore a basis for W.



 $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent and therefore a basis for W. Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$ .



 $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent and therefore a basis for W. Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$ . Let

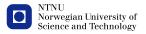
$$\mathbf{v}_{2} = \mathbf{x}_{2} - \operatorname{proj}_{W_{1}} \mathbf{x}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} = \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} \mathbf{1} \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



 $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent and therefore a basis for W. Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$ . Let

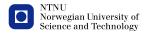
$$\mathbf{v}_{2} = \mathbf{x}_{2} - \operatorname{proj}_{W_{1}} \mathbf{x}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} = \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2'=4\mathbf{v}_2=\left[egin{array}{c}1\\1\\1\end{array}
ight],$$



 $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent and therefore a basis for W. Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$ . Let

$$\mathbf{v}_{2} = \mathbf{x}_{2} - \operatorname{proj}_{W_{1}} \mathbf{x}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} = \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$
  
$$\mathbf{v}_{2}' = 4\mathbf{v}_{2} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and}$$
  
$$W_{2} = \operatorname{Span}\{\mathbf{v}_{1}, \mathbf{v}_{2}\} = \operatorname{Span}\{\mathbf{v}_{1}, \mathbf{v}_{2}\} = \operatorname{Span}\{\mathbf{x}_{1}, \mathbf{x}_{2}\}.$$



Let

$$\mathbf{v}_{3} = \mathbf{x}_{3} - \operatorname{proj}_{W_{2}} \mathbf{x}_{3}$$

$$= x_{3} - \frac{\mathbf{x}_{3} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \frac{\mathbf{x}_{3} \cdot \mathbf{v}_{2}'}{\mathbf{v}_{2}' \cdot \mathbf{v}_{2}'} \mathbf{v}_{2}'$$

$$= \begin{bmatrix} 0\\0\\1\\1\\1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3\\1\\1\\1\\1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0\\-2\\1\\1 \end{bmatrix}$$

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## Solution (cont.)

and 
$$\mathbf{v}_3' = 3\mathbf{v}_3 = \begin{bmatrix} 0\\ -2\\ 1\\ 1 \end{bmatrix}$$
.



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## Solution (cont.)

and 
$$\mathbf{v}_3' = 3\mathbf{v}_3 = \begin{bmatrix} 0\\ -2\\ 1\\ 1\\ 1 \end{bmatrix}$$
. Then  

$$\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\} = \left\{ \begin{bmatrix} 1\\ 1\\ 1\\ 1\\ 1 \end{bmatrix}, \begin{bmatrix} -3\\ 1\\ 1\\ 1\\ 1 \end{bmatrix}, \begin{bmatrix} 0\\ -2\\ 1\\ 1\\ 1 \end{bmatrix} \right\} \text{ is an orthogonal basis}$$
for  $W$ .



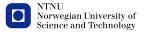


TMA4115 - Calculus 3, Lecture 26, page 32

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#### Theorem 12

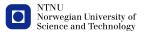
If *A* is an  $m \times n$  matrix with linearly independent columns, then *A* can be factored as A = QR, where



#### Theorem 12

If *A* is an  $m \times n$  matrix with linearly independent columns, then *A* can be factored as A = QR, where

Q is an m × n matrix whose columns form an orthonormal basis for Col(A),



#### Theorem 12

If *A* is an  $m \times n$  matrix with linearly independent columns, then *A* can be factored as A = QR, where

- *Q* is an *m* × *n* matrix whose columns form an orthonormal basis for Col(*A*),
- *R* is an *n* × *n* upper triangular invertible matrix with positive entries on its diagonal.





TMA4115 - Calculus 3, Lecture 26, page 33

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Let  $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$  and let  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be an orthogonal set such that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ .



Let  $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$  and let  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be an orthogonal set such that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ .

Then we have for each  $1 \le k \le p$  that there exist scalars

 $r_{1k}, r_{2k}, \ldots, r_{kk}$  such that  $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \ldots + r_{kk}\mathbf{v}_k$ .



Let  $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$  and let  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be an orthogonal set such that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ .

Then we have for each  $1 \le k \le p$  that there exist scalars  $r_{1k}, r_{2k}, \ldots, r_{kk}$  such that  $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \ldots r_{kk}\mathbf{v}_k$ . We must have that  $r_{kk} \ne 0$  because otherwise  $\mathbf{x}_k$  would be in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k_1}\} =$  Span $\{\mathbf{x}_1, \ldots, \mathbf{x}_{k_1}\}$  which would contradict the assumption that  $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$  is linearly independent.



Let  $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$  and let  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be an orthogonal set such that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  for each  $1 \le k \le p$ .

Then we have for each  $1 \le k \le p$  that there exist scalars

 $r_{1k}, r_{2k}, \ldots, r_{kk}$  such that  $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \ldots + r_{kk}\mathbf{v}_k$ .

We must have that  $r_{kk} \neq 0$  because otherwise  $\mathbf{x}_k$  would be in Span{ $\mathbf{v}_1, \ldots, \mathbf{v}_{k_1}$ } = Span{ $\mathbf{x}_1, \ldots, \mathbf{x}_{k_1}$ } which would contradict the assumption that { $\mathbf{x}_1, \ldots, \mathbf{x}_n$ } is linearly independent.

We may assume that  $r_{kk} > 0$ , because if  $r_{kk} < 0$ , then we can replace  $\mathbf{v}_k$  by  $-\mathbf{v}_k$  and  $r_{kk}$  by  $-r_{kk}$ , and then  $r_{kk} > 0$ .



## Proof of Theorem 12 (cont.)

Let  $Q = [\mathbf{v}_1 \dots \mathbf{v}_n]$  and  $R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$ .



## Proof of Theorem 12 (cont.) Let $Q = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $R = \begin{bmatrix} r_{11} & r_{12} \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$ . Then $QR = [\mathbf{x}_1 \dots \mathbf{x}_n] = A$ .



## Example

# Let us find a *QR* factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .

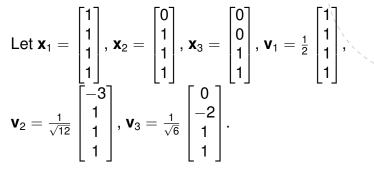


TMA4115 - Calculus 3, Lecture 26, page 35



TMA4115 - Calculus 3, Lecture 26, page 36

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## Solution (cont.)

Then  $A = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$ , and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is an orthonormal set such that

$$\begin{aligned} \mathbf{x}_1 &= 2\mathbf{v}_1 \\ \mathbf{x}_2 &= \frac{3}{2}\mathbf{v}_1 + \frac{\sqrt{3}}{2}\mathbf{v}_2 \\ \mathbf{x}_3 &= \mathbf{v}_1 + \frac{1}{\sqrt{3}}\mathbf{v}_2 + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{v}_3 \end{aligned}$$



## Solution (cont.)

So if we let 
$$Q = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{12}}{\sqrt{12}} & 0\\ \frac{1}{2} & \frac{\sqrt{12}}{\sqrt{12}} & \frac{\sqrt{6}}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$
 and  $R = \begin{bmatrix} 2 & \frac{3}{2} & 1\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}}\\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$ , then  $QR = A$  is a  $QR$  factorization of  $A$ .

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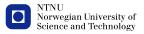


## Problem 6 from June 2010

Let *A* be the following matrix; find a basis for each of the spaces Nul(*A*), Col(*A*), (Col(*A*))<sup> $\perp$ </sup>, and Row(*A*).

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix}$$

Find the orthogonal projection of  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  on to Col(*A*).





TMA4115 - Calculus 3, Lecture 26, page 40

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We start by reducing A to its reduced echelon form.

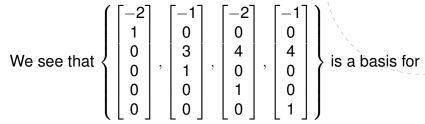


TMA4115 - Calculus 3, Lecture 26, page 40

We start by reducing A to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 2 & -6 & -8 & -8 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

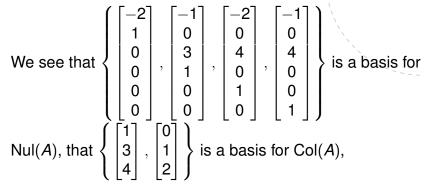




Nul(A),

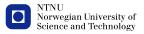


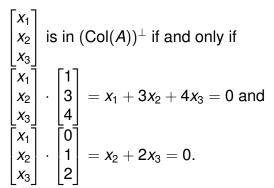
TMA4115 - Calculus 3, Lecture 26, page 41





and that  $\left\{ \begin{array}{c|c} 2 \\ 0 \\ 1 \\ 2 \\ 2 \\ \end{array}, \begin{array}{c} 0 \\ 1 \\ -3 \\ -4 \\ \end{array} \right\}$  is a basis for Row(A).







$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is in } (\operatorname{Col}(A))^{\perp} \text{ if and only if}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = x_1 + 3x_2 + 4x_3 = 0 \text{ and}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = x_2 + 2x_3 = 0.$$
We reduce the coefficient matrix of the system

$$x_1 + 3x_2 + 4x_3 = 0$$

$$x_2 + 2x_3 = 0$$
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to its reduced echelon form.

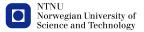
sitv of

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$



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$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$
  
We see that  $(\operatorname{Col}(A))^{\perp} = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ , so  $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$  is a basis for  $(\operatorname{Col}(A))^{\perp}$ .



The orthogonal projection of  $\begin{bmatrix} 0\\0\\1 \end{bmatrix}$  on to  $(Col(A))^{\perp}$  is

$$\begin{bmatrix}
0 \\
0 \\
-2 \\
1
\end{bmatrix}
\begin{bmatrix}
2 \\
-2 \\
-2 \\
1
\end{bmatrix}
\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}
\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}
= \frac{1}{9}\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}$$



The orthogonal projection of  $\begin{bmatrix} 0\\0\\1 \end{bmatrix}$  on to  $(\operatorname{Col}(A))^{\perp}$  is

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#### Problem 5 from June 2011

Let *V* be the column space of the matrix

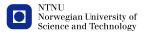
$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

and let

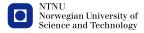
$$\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$
 .

Find the nearest point in V to **b** (that is, the orthogonal projection of **b** on to V).

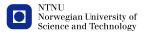




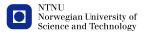
TMA4115 - Calculus 3, Lecture 26, page 47



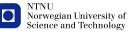
$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

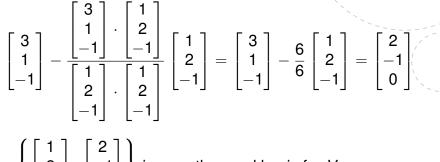


$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
  
We see that  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$  is a basis for V.

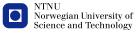


$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
  
We see that  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$  is a basis for *V*. We then find  
an orthogonal basis for *V* by using the Gram-Schmidt  
process on  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ .

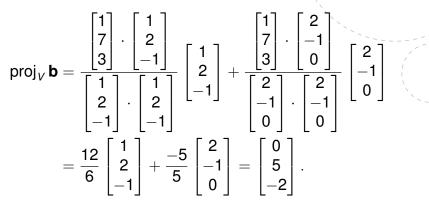


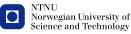


so 
$$\left\{ \begin{bmatrix} 1\\ 2\\ -1 \end{bmatrix}, \begin{bmatrix} 2\\ -1\\ 0 \end{bmatrix} \right\}$$
 is an orthogonal basis for *V*.



#### We then have that





### Problem 5 from December 2010

Let  $V \subseteq \mathbb{R}^4$  be the solution space of the linear system

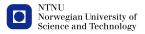
$$x + y - z + w = 0$$
$$x + 2y - 2z + w = 0$$

Find an orthogonal basis for V.

- Find the orthogonal projection of  $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  on to V.
- Sind an orthogonal basis for ℝ<sup>4</sup> in which the first two first basis vectors are the once we found in (1).



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TMA4115 - Calculus 3, Lecture 26, page 51

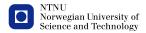
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We start by reducing the coefficient matrix of the system to its reduced echelon form.



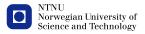
We start by reducing the coefficient matrix of the system to its reduced echelon form.

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$



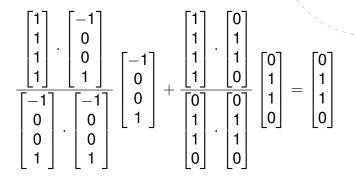
We start by reducing the coefficient matrix of the system to its reduced echelon form.

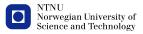
$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}^{-1}$$
  
We see that  $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$  is a basis for *V* and that it is orthogonal.



0

The orthogonal projection of  $\mathbf{b}$  on to V is



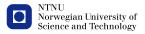


Let *A* be the coefficient matrix of the system.



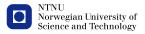
Let *A* be the coefficient matrix of the system. Then  $\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ 

$$V^{\perp} = (\operatorname{Nul}(A))^{\perp} = \operatorname{Row}(A) = \operatorname{Span} \left\{ egin{array}{c} 0 \\ 0 \\ 1 \end{bmatrix}, egin{array}{c} 1 \\ -1 \\ 0 \end{bmatrix} 
ight\}$$



Let *A* be the coefficient matrix of the system. Then  $\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ 

$$V^{\perp} = (\operatorname{Nul}(A))^{\perp} = \operatorname{Row}(A) = \operatorname{Span} \left\{ \begin{bmatrix} 0\\0\\1 \end{bmatrix}, \begin{bmatrix} 1\\-1\\0 \end{bmatrix} \right\}, \text{ so} \\ \left\{ \begin{bmatrix} -1\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\-1\\0 \end{bmatrix} \right\} \text{ is an orthogonal basis for } \right\}$$



#### Plan for next week



TMA4115 - Calculus 3, Lecture 26, page 54

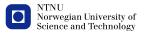
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#### Plan for next week

Wednesday we shall look at

- least-squares problems,
- applications to linear models.

Sections 6.5–6.6 in "Linear Algebras and Its Applications" (pages 360–375).



## Plan for next week

Wednesday we shall look at

- least-squares problems,
- applications to linear models.

Sections 6.5–6.6 in "Linear Algebras and Its Applications" (pages 360–375).

Thursday we shall introduce and study

- symmetric matrices,
- quadratic forms.

Sections 7.1–7.2 in "Linear Algebras and Its Applications" (pages 393–407).

