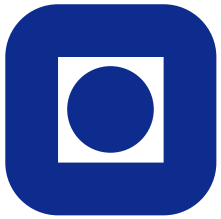

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# TMA4115 - Calculus 3

## Lecture 22, April 4

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

# Yesterday's lecture

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Yesterday we introduced and studied

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,

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Yesterday we introduced and studied

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix.

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# Today's lecture

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# Today's lecture

Today we shall look at

- *diagonal* matrices,

# Today's lecture

Today we shall look at

- *diagonal* matrices,
- how to *diagonalizable* a matrix.

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# Eigenvectors, eigenvalues and eigenspaces

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Let $A$ be an $n \times n$ matrix.

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Let $A$ be an $n \times n$ matrix. Recall that

- An *eigenvalue* of $A$ is a scalar $\lambda$ such that the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution.

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- If $\lambda$ is an eigenvalue of $A$, then a nontrivial solution to the equation $A\mathbf{x} = \lambda\mathbf{x}$ is called an *eigenvector (of A) corresponding to* $\lambda$.

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- If $\lambda$ is an eigenvalue of $A$, then the set of solutions to the equation $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenspace (of A) corresponding to* $\lambda$.

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# Finding a basis of an eigenspace

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# Finding a basis of an eigenspace

Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$.

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Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. Recall that the eigenspace of $A$ corresponding to $\lambda$ is equal to $\text{Nul}(A - \lambda I_n)$, so we can find a basis for the eigenspace of $A$ corresponding to $\lambda$ by:

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1. row reducing $A - \lambda I_n$ to its reduced echelon form,
2. write the solutions to the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ (which is equivalent to the equation $A\mathbf{x} - \mathbf{0}$) as a linear combinations of vectors using the free variables as parameters.

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Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. Recall that the eigenspace of $A$ corresponding to $\lambda$ is equal to $\text{Nul}(A - \lambda I_n)$, so we can find a basis for the eigenspace of $A$ corresponding to $\lambda$ by:

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3. Then these vectors form a basis of the eigenspace of $A$ corresponding to $\lambda$.

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# Finding the eigenvalues of a matrix

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# Finding the eigenvalues of a matrix

Let $A$ be an $n \times n$ matrix. Recall that a scalar $\lambda$ is an eigenvalue of $A$ if and only if it is a zero of the *characteristic polynomial* $\det(A - \lambda I_n)$ of $A$.

# Example

Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$.

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# Example

Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$.

The characteristic polynomial of $A$ is

$$\begin{aligned} \det(A - \lambda I_4) &= \begin{vmatrix} 1-\lambda & -3 & 6 & 0 \\ 0 & -2-\lambda & 1 & 1 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & 0 & 7-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda)(-\lambda)(7-\lambda). \end{aligned}$$

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# Example

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It follows that the eigenvalues of $A$ are 1, $-2$, 0 and 7.

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# The eigenvalues of a triangular matrix

**Theorem 1**

The eigenvalues of a triangular matrix are the entries on its main diagonal.

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# Example

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# Example

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -5 & 6 & -3 \\ 5 & 2 & 11 \end{bmatrix}$, $D = \begin{bmatrix} 4 & -8 & 4 \\ 0 & 6 & 4 \\ 0 & 0 & 8 \end{bmatrix}$ and

$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

# Example

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -5 & 6 & -3 \\ 5 & 2 & 11 \end{bmatrix}$, $D = \begin{bmatrix} 4 & -8 & 4 \\ 0 & 6 & 4 \\ 0 & 0 & 8 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Then $P$ is invertible and $P^{-1}AP = D$.

# Example (cont.)

Then $D - \lambda I_n = P^{-1}AP - \lambda P^{-1} I_n P = P^{-1}(A - \lambda I_n)P$.

# Example (cont.)

Then $D - \lambda I_n = P^{-1}AP - \lambda P^{-1} I_n P = P^{-1}(A - \lambda I_n)P$. So

$$\begin{aligned}
\det(D - \lambda I_n) &= \det(P^{-1}(A - \lambda I_n)P) \\
&= \det(P^{-1}) \det(A - \lambda I_n) \det(P) \\
&= \det(P^{-1}) \det(P) \det(A - \lambda I_n) \\
&= \det(P^{-1} P) \det(A - \lambda I_n) \\
&= \det(I_n) \det(A - \lambda I_n) \\
&= \det(A - \lambda I_n).
\end{aligned}$$

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# Example (cont.)

Then $D - \lambda I_n = P^{-1}AP - \lambda P^{-1} I_n P = P^{-1}(A - \lambda I_n)P$. So

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It follows that $A$ and $D$ have the same eigenvalues.

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&= \det(I_n) \det(A - \lambda I_n) \\
&= \det(A - \lambda I_n).
\end{aligned}$$

It follows that $A$ and $D$ have the same eigenvalues. So the eigenvalues of $A$ are 4,6 and 8.

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# Similarity

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Let $A$ and $B$ be $n \times n$ matrices.

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- $A$ is *similar* to $B$ if there is an invertible matrix $P$ such that $P^{-1}AP = B$.
- If $A$ is similar to $B$, then $B$ is similar to $A$ (because if $P^{-1}AP = B$, and we let $Q = P^{-1}$, then $Q^{-1}BQ = PBP^{-1} = PP^{-1}BP^{-1}P = A$).

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- If $A$ is similar to $B$ (and thus $B$ is similar to $A$), then we say that $A$ and $B$ *are similar*.

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# The characteristic polynomials of similar matrices

## Theorem 4

If two matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicity).

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The *multiplicity* of an eigenvalue $\lambda_k$ of an $n \times n$ matrix $A$ is the maximal $m$ such that $(\lambda - \lambda_k)^m$ is a factor of $\det(A - \lambda I_n)$.

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# Proof of Theorem 4

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# Proof of Theorem 4

If $A$ and $B$ are similar, then there is an invertible matrix $P$ such that $P^{-1}AP = B$. In that case,
$B - \lambda I_n = P^{-1}AP - \lambda P^{-1} I_n P = P^{-1}(A - \lambda I_n)P$ and

$$\begin{aligned}
\det(B - \lambda I_n) &= \det(P^{-1}(A - \lambda I_n)P) \\
&= \det(P^{-1}) \det(A - \lambda I_n) \det(P) \\
&= \det(P^{-1}) \det(P) \det(A - \lambda I_n) \\
&= \det(P^{-1}P) \det(A - \lambda I_n) \\
&= \det(I_n) \det(A - \lambda I_n) \\
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\end{aligned}$$

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# Finding eigenvalues via the QR algorithm

The previous theorem can be used to calculate the eigenvalues of a square matrix.

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# Finding eigenvalues via the QR algorithm

The previous theorem can be used to calculate the eigenvalues of a square matrix.

- If $A$ is an $n \times n$-matrix, then there exists under certain conditions a sequence $A_1, A_2, \ldots$ of $n \times n$-matrix such that each $A_k$ is similar to $A$, and such that $A_k$ converges to a triangular matrix $B$.

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- Since each $A_k$ is similar to $A$, the eigenvalues of each $A_k$, and hence of $B$, are the same as the eigenvalues of $A$.

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# Finding eigenvalues via the QR algorithm

The previous theorem can be used to calculate the eigenvalues of a square matrix.

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- Since each $A_k$ is similar to $A$, the eigenvalues of each $A_k$, and hence of $B$, are the same as the eigenvalues of $A$.
- So the diagonal entries of $A_k$ converge to the eigenvalues of $A$.

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# Diagonalizable matrices

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- A *diagonal* matrix is a square matrix whose entries outside the main diagonal are all zero.

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- If $D$ is a diagonal matrix, then $\mathbf{e}_i$ is an eigenvector of $D$ corresponding to the $i$"th entry on the main diagonal.

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- If $D$ is a diagonal matrix, then $\mathbf{e}_i$ is an eigenvector of $D$ corresponding to the $i$'th entry on the main diagonal.
- A matrix $A$ is *diagonalizable* if it is similar to a diagonal matrix (that is, if $A = PDP^{-1}$ for some invertible matrix $P$ and some diagonal matrix $D$).

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# The diagonalization Theorem

## Theorem 5

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

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In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are $n$ linearly independent eigenvectors of $A$, in which case the diagonal entries of $D$ are eigenvalues of $A$ corresponding, respectively, to the eigenvectors in $P$.

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# Proof of Theorem 5

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Suppose that $P = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is an invertible matrix,

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}, \text{ and } A = PDP^{-1}.$$

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Then

$$A\mathbf{v}_k = PDP^{-1}\mathbf{v}_k = PD\mathbf{e}_k = P\lambda_k\mathbf{e}_k = \lambda_k\mathbf{v}_k,$$

so $\lambda_k$ is an eigenvalue of $A$, and $\mathbf{v}_k$ is a corresponding eigenvector.

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so $\lambda_k$ is an eigenvalue of $A$, and $\mathbf{v}_k$ is a corresponding eigenvector. Since $P$ is invertible, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, so $A$ has $n$ linearly independent eigenvectors.

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# Proof of Theorem 5 (cont.)

Conversely, suppose $A$ is an $n \times n$ matrix which has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

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# Proof of Theorem 5 (cont.)

Conversely, suppose $A$ is an $n \times n$ matrix which has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let for each $k$, $\lambda_k$ be the eigenvalue corresponding to $\mathbf{v}_k$.

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# Proof of Theorem 5 (cont.)

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$$P = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

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$$P = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Then we have for each $k$ that

$$PD\mathbf{e}_k = P\lambda_k\mathbf{e}_k = \lambda_k\mathbf{v}_k$$

and that

$$AP\mathbf{e}_k = A\mathbf{v}_k = \lambda_k\mathbf{v}_k.$$

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# Proof of Theorem 5 (cont.)

Conversely, suppose $A$ is an $n \times n$ matrix which has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let for each $k$, $\lambda_k$ be the eigenvalue corresponding to $\mathbf{v}_k$. Let

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Then we have for each $k$ that

$$PD\mathbf{e}_k = P\lambda_k\mathbf{e}_k = \lambda_k\mathbf{v}_k$$

and that

$$AP\mathbf{e}_k = A\mathbf{v}_k = \lambda_k\mathbf{v}_k.$$

It follows that $PD = AP$.

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# Proof of Theorem 5 (cont.)

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, it follows that $P$ is invertible, so $A = PDP^{-1}$.

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# How to diagonalize a matrix (if it is diagonalizable)

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4. Construct $D$ from the corresponding eigenvalues.

# How to diagonalize a matrix (if it is diagonalizable)

1. Find the eigenvalues of $A$.
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3. Construct $P$ from the linearly independent eigenvectors of $A$.
4. Construct $D$ from the corresponding eigenvalues.
5. Then $A = PDP^{-1}$.

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# Example

Let us determine if $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$ is diagonalizable and diagonalize it if it is and find an expression for $A^n$.

# Solution

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# Solution

The characteristic polynomial of $A$ is

$$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 2 \\ -3 & 5-\lambda \end{vmatrix} = -\lambda(5-\lambda) + 6 = \lambda^2 - 5\lambda + 6.$$

The zeros of $\lambda^2 - 5\lambda + 6$ are $\lambda = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$, so the eigenvalues of $A$ are 2 and 3.

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The zeros of $\lambda^2 - 5\lambda + 6$ are $\lambda = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$, so the eigenvalues of $A$ are 2 and 3.

$A - 2I_2 = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

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The zeros of $\lambda^2 - 5\lambda + 6$ are $\lambda = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$, so the eigenvalues of $A$ are 2 and 3.

$A - 2I_2 = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

$A - 3I_2 = \begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3.

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# Solution (cont.)

Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are two linearly independent eigenvectors of $A$, it follows that $A$ is diagonalizable.

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Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are two linearly independent eigenvectors of $A$, it follows that $A$ is diagonalizable.

Let $P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

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# Solution (cont.)

Since $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ are two linearly independent eigenvectors of $A$, it follows that $A$ is diagonalizable.

Let $P = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}$ and $D = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$.

Then $AP = \begin{bmatrix}0 & 2\\-3 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}2 & 6\\2 & 9\end{bmatrix}$

and $PD = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}2 & 6\\2 & 9\end{bmatrix}$.

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# Solution (cont.)

$P$ is invertible since $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ are linearly independent, so it follows that $A = PDP^{-1}$ and that

$$\begin{aligned} A^n = (PDP^{-1})^n &= \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}2^n & 0\\0 & 3^n\end{bmatrix}\begin{bmatrix}3 & -2\\-1 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}3\cdot 2^n & -2^{n+1}\\-3^n & 3^n\end{bmatrix} \\ &= \begin{bmatrix}3\cdot 2^n - 2\cdot 3^n & -2^{n+1} + 2\cdot 3^n\\3\cdot 2^n - 3^{n+1} & -2^{n+1} + 3^{n+1}\end{bmatrix}. \end{aligned}$$

# Matrices with $n$ distinct eigenvalues are diagonalizable

**Theorem 6**

An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

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# Proof of Theorem 6

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# Proof of Theorem 6

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the $n$ distinct eigenvalues.

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# Proof of Theorem 6

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the $n$ distinct eigenvalues. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent by Theorem 2.

# Proof of Theorem 6

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the $n$ distinct eigenvalues. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent by Theorem 2. So it follows from Theorem 5 that $A$ is diagonalizable.

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# Matrices with fewer than $n$ distinct eigenvalues

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# Matrices with fewer than $n$ distinct eigenvalues

## Theorem 7

Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

# Matrices with fewer than $n$ distinct eigenvalues

## Theorem 7

Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

1. For $1 \leq k \leq p$, the dimension of the eigenspace for $\lambda_k$ is less than or equal to the multiplicity of the eigenvalue $\lambda_k$.

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# Matrices with fewer than $n$ distinct eigenvalues

## Theorem 7

2. $A$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals $n$, and this happens if and only if the characteristic polynomial factors completely into linear factors and the dimension of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.

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# Matrices with fewer than $n$ distinct eigenvalues

## Theorem 7

2. $A$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals $n$, and this happens if and only if the characteristic polynomial factors completely into linear factors and the dimension of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.
3. If $A$ is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to $\lambda_k$ for each $k$, then the total collection of vectors in $\mathcal{B}_1, \ldots, \mathcal{B}_k$ forms an eigenvector basis for $\mathbb{R}^n$.

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# Problem 7 from the exam from June 2012

1. Find all the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix}$.
2. Find a basis for each eigenspace of $A$. Is $A$ diagonalisable?

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# Solution

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# Solution

The characteristic polynomial of $A$ is

$$\det(A-\lambda I_3) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big)$$

$$= (2-\lambda)(\lambda^2-5\lambda+6).$$

# Solution

The characteristic polynomial of $A$ is

$$\begin{aligned}\det(A-\lambda I_3) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big) \\ &= (2-\lambda)(\lambda^2-5\lambda+6).\end{aligned}$$

The zeros of $\lambda^2-5\lambda+6$ are $\lambda = \frac{5\pm\sqrt{5^2-4\cdot 6}}{2} = \frac{5\pm 1}{2}$,

# Solution

The characteristic polynomial of $A$ is

$$\begin{aligned}\det(A-\lambda I_3) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big) \\ &= (2-\lambda)(\lambda^2-5\lambda+6).\end{aligned}$$

The zeros of $\lambda^2-5\lambda+6$ are $\lambda = \frac{5\pm\sqrt{5^2-4\cdot 6}}{2} = \frac{5\pm 1}{2}$, so the eigenvalues of $A$ are 2 and 3 (notice that 2 has multiplicity 2).

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# Solution (cont.)

To find a basis for the eigenspace of $A$ corresponding to the eigenvalue 2, we reduce $A - 2I_3$ to its reduced echelon form.

# Solution (cont.)

To find a basis for the eigenspace of $A$ corresponding to the eigenvalue 2, we reduce $A - 2I_3$ to its reduced echelon form.

$$A - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the general solution to the equation $(A - 2I_3)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

where $x_1$ and $x_3$ are free parameters.

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# Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $A$ corresponding to the eigenvalue 2.

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# Solution (cont.)

To find a basis for the eigenspace of $A$ corresponding to the eigenvalue 3, we reduce $A - 3I_3$ to its reduced echelon form.

# Solution (cont.)

To find a basis for the eigenspace of $A$ corresponding to the eigenvalue 3, we reduce $A - 3I_3$ to its reduced echelon form.

$$A - 3I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the general solution to the equation $(A - 3I_3)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where $x_3$ is a free parameter.

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# Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $A$ corresponding to the eigenvalue 3.

# Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $A$ corresponding to the eigenvalue 3.

$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ (because if it was, then it would be an eigenvector of $A$ corresponding to both 2 and 3),

# Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $A$ corresponding to the eigenvalue 3.

$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ (because if it was, then it would be an eigenvector of $A$ corresponding to both 2 and 3),

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

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# Solution (cont.)

It follows that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $A$ corresponding to the eigenvalue 3.

$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ (because if it was, then it would be an eigenvector of $A$ corresponding to both 2 and 3), so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent. So $A$ has 3 linearly independent eigenvectors, and it is therefore diagonalizable.

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# Problem 11 from August 2004

Let $A$ be a diagonalizable matrix where every eigenvalue is either 1 or $-1$. Show that $A^{-1} = A$.

# Solution

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# Solution

If $A$ be a diagonalizable matrix where every eigenvalue is either 1 or $-1$, then $A = P^{-1}DP$ where $P$ is an invertible matrix and $D$ is a diagonal matrix where all the entries on the diagonal is either 1 or $-1$.

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# Solution

If $A$ be a diagonalizable matrix where every eigenvalue is either 1 or $-1$, then $A = P^{-1}DP$ where $P$ is an invertible matrix and $D$ is a diagonal matrix where all the entries on the diagonal is either 1 or $-1$.
$D^2 = I$,

# Solution

If $A$ be a diagonalizable matrix where every eigenvalue is either 1 or $-1$, then $A = P^{-1}DP$ where $P$ is an invertible matrix and $D$ is a diagonal matrix where all the entries on the diagonal is either 1 or $-1$.
$D^2 = I$, so $D$ is invertible and $D^{-1} = D$.

# Solution

If $A$ be a diagonalizable matrix where every eigenvalue is either 1 or $-1$, then $A = P^{-1}DP$ where $P$ is an invertible matrix and $D$ is a diagonal matrix where all the entries on the diagonal is either 1 or $-1$.
$D^2 = I$, so $D$ is invertible and $D^{-1} = D$. It follows that $A$ is invertible and that
$A^{-1} = (P^{-1}DP)^{-1} = P^{-1}D^{-1}(P^{-1})^{-1} = P^{-1}DP = A$.

# Problem 7 from December 2005

Let $A$ and $B$ be two $n \times n$-matrices.

1. Show that if 0 is an eigenvalue of $AB$, then 0 is an eigenvalue of $BA$.
2. Show that if $B$ is invertible and $\lambda$ is an eigenvalue of $AB$, then $\lambda$ is an eigenvalue of $BA$.

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# Solution

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# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$.

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# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$,

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# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of $BA$.

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# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of $BA$.

Suppose now that $\lambda$ is a nonzero eigenvalue of $AB$, and let $\mathbf{x}$ be a corresponding eigenvector.

# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of $BA$.

Suppose now that $\lambda$ is a nonzero eigenvalue of $AB$, and let $\mathbf{x}$ be a corresponding eigenvector. Then $AB\mathbf{x} = \lambda\mathbf{x}$,

# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of $BA$.

Suppose now that $\lambda$ is a nonzero eigenvalue of $AB$, and let $\mathbf{x}$ be a corresponding eigenvector. Then $AB\mathbf{x} = \lambda\mathbf{x}$, so if we let $\mathbf{z} = B\mathbf{x}$, then $BA\mathbf{z} = BAB\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x} = \lambda\mathbf{z}$.

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# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of $BA$.

Suppose now that $\lambda$ is a nonzero eigenvalue of $AB$, and let $\mathbf{x}$ be a corresponding eigenvector. Then $AB\mathbf{x} = \lambda\mathbf{x}$, so if we let $\mathbf{z} = B\mathbf{x}$, then $BA\mathbf{z} = BAB\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x} = \lambda\mathbf{z}$. The vector $\mathbf{z}$ must be nonzero, because otherwise $\lambda\mathbf{x} = AB\mathbf{x} = A\mathbf{z} = 0$ which would contradict the fact that $\lambda \neq 0$ and $\mathbf{x} \neq \mathbf{0}$.

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# Solution

Since the eigenvalues of an $n \times n$-matrix $C$ are the zeros of the characteristic polynomial $\det(C - \lambda I_n)$, it follows that 0 is an eigenvalue of $C$ if and only if $\det(C) = 0$. So if 0 is an eigenvalue of $AB$, then $\det(BA) = \det(AB) = 0$, and then 0 is an eigenvalue of $BA$.

Suppose now that $\lambda$ is a nonzero eigenvalue of $AB$, and let $\mathbf{x}$ be a corresponding eigenvector. Then $AB\mathbf{x} = \lambda\mathbf{x}$, so if we let $\mathbf{z} = B\mathbf{x}$, then $BA\mathbf{z} = BAB\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x} = \lambda\mathbf{z}$. The vector $\mathbf{z}$ must be nonzero, because otherwise $\lambda\mathbf{x} = AB\mathbf{x} = A\mathbf{z} = 0$ which would contradict the fact that $\lambda \neq 0$ and $\mathbf{x} \neq \mathbf{0}$.

So $\mathbf{z}$ is an eigenvector of $BA$, and $\lambda$ is the corresponding eigenvalue.

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# Review quiz

# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.

# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.
2. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.

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# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.
2. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
3. If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.

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# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.
2. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
3. If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.
4. Each eigenvector of $A$ is also an eigenvector of $A^2$.

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# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.
2. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
3. If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.
4. Each eigenvector of $A$ is also an eigenvector of $A^2$.
5. If $A$ is an $n \times n$ diagonalizable matrix, then each vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of $A$.

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# Review quiz

Let $A$ be a square matrix. Decide for each of the following statements, whether the statement is true or false.

1. The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.
2. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
3. If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.
4. Each eigenvector of $A$ is also an eigenvector of $A^2$.
5. If $A$ is an $n \times n$ diagonalizable matrix, then each vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of $A$.
6. If $A$ contains a row or column of zeros, then 0 is an eigenvalue of $A$.

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# Solution

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# Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of $A$.

# Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of $A$.
2. False, if an eigenspace has dimension higher than 1, then it contains two linearly independent eigenvectors.

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# Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of $A$.
2. False, if an eigenspace has dimension higher than 1, then it contains two linearly independent eigenvectors.
3. True, because $AB = A(BA)A^{-1}$.

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# Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of $A$.
2. False, if an eigenspace has dimension higher than 1, then it contains two linearly independent eigenvectors.
3. True, because $AB = A(BA)A^{-1}$.
4. True, if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = A\lambda\mathbf{x} = \lambda^2\mathbf{x}$.

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# Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of $A$.
2. False, if an eigenspace has dimension higher than 1, then it contains two linearly independent eigenvectors.
3. True, because $AB = A(BA)A^{-1}$.
4. True, if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = A\lambda\mathbf{x} = \lambda^2\mathbf{x}$.
5. True, if $A$ is an $n \times n$ diagonalizable matrix, then it has $n$ linearly independent eigenvectors, and these eigenvectors will then form a basis of $\mathbb{R}^n$.

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# Solution

1. False, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\mathbf{v}_1 + \mathbf{v}_2$ cannot be an eigenvector of $A$.
2. False, if an eigenspace has dimension higher than 1, then it contains two linearly independent eigenvectors.
3. True, because $AB = A(BA)A^{-1}$.
4. True, if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = A\lambda\mathbf{x} = \lambda^2\mathbf{x}$.
5. True, if $A$ is an $n \times n$ diagonalizable matrix, then it has $n$ linearly independent eigenvectors, and these eigenvectors will then form a basis of $\mathbb{R}^n$.
6. True, because 0 is an eigenvalue of $A$ if and only if $\det(A) = 0$, and if $A$ contains a row or column of zeros, then $\det(A) = 0$.

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# Plan for next week

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# Plan for next week

Next week we shall study

- real matrices with complex eigenvalues,
- systems of first order differential equations.

Sections 5.5 and 5.7 in “Linear Algebras and Its Applications” (pages 295-301, and 311-319), and Section 4.2 in “Second-Order Equations” (pages xlv-xlix).

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