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TMA4115 - Calculus 3

Lecture 15, March 6

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of last week's lecture

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Review of last week's lecture

Last week we looked at

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Review of last week's lecture

Last week we looked at

- how to *add* and *multiply* matrices,

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- how to *add* and *multiply* matrices,
- *invertible* matrices and their *inverses*,

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Review of last week's lecture

Last week we looked at

- how to *add* and *multiply* matrices,
- *invertible* matrices and their *inverses*,
- *the invertible matrix theorem*.

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Today’s lecture

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Today's lecture

Today we shall introduce and study *determinants*.

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The inverse of an invertible 2×2 matrix

Recall the following result from last week:

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The inverse of an invertible 2×2 matrix

Recall the following result from last week:

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The number $ad - bc$ is called the *determinant* of A.

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Determinant

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Determinant

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- A square matrix A is invertible if and only if $\det(A) \neq 0$.

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- The determinant can be used to give an explicit formula for the inverse of an invertible matrix.
- $\det(AB) = \det(A)\det(B)$.

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- The determinant is a value associated with a square matrix.
- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- The determinant can be used to give an explicit formula for the inverse of an invertible matrix.
- $\det(AB) = \det(A)\det(B)$.
- The absolute value of the determinant gives the scale factor by which area or volume is multiplied under the associated linear transformation.

The definition of the determinant

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For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A.

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Definition

The *determinant* of a 1×1 matrix $A = [a]$ is $\det(A) = a$.

The definition of the determinant

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A.

Definition

The *determinant* of a 1×1 matrix $A = [a]$ is $\det(A) = a$.
For $n \geq 2$, the *determinant* of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

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Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$$

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$\begin{aligned} \det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) \\ &= 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \end{aligned}$$

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Cofactor expansions

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Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

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Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

Proof of Theorem 1

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We will prove the theorem by induction over n.

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Assume that $k > 1$ and that the theorem is true for $n = k - 1$.

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Proof of Theorem 1

We will prove the theorem by induction over n.
The theorem is obviously true for $n = 1$.
Assume that $k > 1$ and that the theorem is true for $n = k - 1$.
Let A be a $k \times k$ matrix, let h be an integer between 1 and k, and let i be an integer between 2 and k.

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Proof of Theorem 1

We will prove the theorem by induction over n.
The theorem is obviously true for $n = 1$.
Assume that $k > 1$ and that the theorem is true for $n = k - 1$.
Let A be a $k \times k$ matrix, let h be an integer between 1 and k, and let i be an integer between 2 and k. Then A_{1h} is a $(k-1) \times (k-1)$ matrix, so

$$\det A_{1h} = \sum_{j=1}^{h-1} (-1)^{i+j} a_{ij} \det(A_{1h})_{ij} + \sum_{j=h+1}^{k} (-1)^{i+j} a_{ij} \det(A_{1h})_{ij}$$

by the induction assumption.

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Proof of Theorem 1 (cont.)

We furthermore have that if j is an integer between 1 and k different from h, then $(A_{1h})_{ij} = (A_{ij})_{1h}$.

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$$\sum_{j=1}^{k} a_{ij} C_{ij}$$

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$$\sum_{j=1}^{k} a_{ij} C_{ij} = \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \det A_{ij}$$

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Proof of Theorem 1 (cont.)

We furthermore have that if j is an integer between 1 and k different from h, then $(A_{1h})_{ij} = (A_{ij})_{1h}$. Thus

$$\begin{aligned}\sum_{j=1}^{k} a_{ij} C_{ij} &= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \det A_{ij} \\ &= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{1+h} a_{1h} \det(A_{ij})_{1h} \right. \\ &\qquad\qquad \left. + \sum_{h=j+1}^{k} (-1)^{1+h} a_{1h} \det(A_{ij})_{1h} \right)\end{aligned}$$

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Proof of Theorem 1 (cont.)

$$= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{1+h} a_{1h} \det(A_{1h})_{ij} \right.$$

$$\left. + \sum_{h=j+1}^{k} (-1)^{1+h} a_{1h} \det(A_{1h})_{ij} \right)$$

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Proof of Theorem 1 (cont.)

$$= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{1+h} a_{1h} \det(A_{1h})_{ij} + \sum_{h=j+1}^{k} (-1)^{1+h} a_{1h} \det(A_{1h})_{ij} \right)$$

$$= \sum_{h=1}^{k} (-1)^{1+h} a_{1h} \left(\sum_{j=1}^{h-1} (-1)^{i+j} a_{ij} \det(A_{1h})_{ij} + \sum_{j=h+1}^{k} (-1)^{i+j} a_{+j} \det(A_{1h})_{ij} \right)$$

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Proof of Theorem 1 (cont.)

$$= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \det A_{1h}$$

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$$= \det A.$$

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Similarly, if j is an integer between 1 and k, then

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Proof of Theorem 1 (cont.)

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Proof of Theorem 1 (cont.)

$$= \sum_{j=1}^{k} (-1)^{i+j} a_{ij} \det A_{1h}$$
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Similarly, if j is an integer between 1 and k, then

$$\sum_{i=1}^{k} a_{ij} C_{ij} = \sum_{i=1}^{k} (-1)^{i+j} a_{ij} \det A_{ij}$$
$$= \sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \det A_{ij} + a_{jj} \det A_{jj} +$$
$$\sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \det A_{ij}$$

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Proof of Theorem 1 (cont.)

$$= \sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right.$$

$$\left. + \sum_{h=j+1}^{k} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right)$$

$$+ a_{jj} \det A_{jj}$$

$$+ \sum_{i=j+1}^{k} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{ij})_{jh} \right.$$

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Proof of Theorem 1 (cont.)

$$= \sum_{i=1}^{j-1} (-1)^{i+j} a_{ij} \left(\sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det(A_{jh})_{ij} + \sum_{h=j+1}^{k} (-1)^{j+h} a_{jh} \det(A_{jh})_{ij} \right)$$

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Proof of Theorem 1 (cont.)

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Proof of Theorem 1 (cont.)

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Proof of Theorem 1 (cont.)

$$= \sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det A_{jh} + a_{jj} \det A_{jj} + \sum_{h=j+1}^{k} (-1)^{j+h} a_{hj} \det A_{jh}$$

$$= \sum_{h=1}^{k} (-1)^{j+h} a_{jh} \det A_{jh} = \det A.$$

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Proof of Theorem 1 (cont.)

$$= \sum_{h=1}^{j-1} (-1)^{j+h} a_{jh} \det A_{jh} + a_{jj} \det A_{jj} + \sum_{h=j+1}^{k} (-1)^{j+h} a_{hj} \det A_{jh}$$

$$= \sum_{h=1}^{k} (-1)^{j+h} a_{jh} \det A_{jh} = \det A.$$

Thus it follows by induction that the theorem is true for all n.

Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

$$\det(A)$$

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

$$\det(A) = (-1)^{2+3}(-1)\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix}$$

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Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ by using a cofactor expansion across the third column.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

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Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

$$\det(A) = 1 \cdot \begin{vmatrix} 2 & 5 & -1 \\ 0 & -4 & 5 \\ 0 & 0 & -3 \end{vmatrix}$$

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$$= 1 \cdot 2 \cdot (-4) \det([-3])$$

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$$= 1 \cdot 2 \cdot (-4) \det([-3]) = 1 \cdot 2 \cdot (-4) \cdot (-3)$$

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Example

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 2 & 5 & -1 \\ 0 & -4 & 5 \\ 0 & 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} -4 & 5 \\ 0 & -3 \end{vmatrix} \\ &= 1 \cdot 2 \cdot (-4) \det([-3]) = 1 \cdot 2 \cdot (-4) \cdot (-3) \\ &= 24. \end{aligned}$$

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

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Proof of Theorem 2

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Proof of Theorem 2

We will prove the theorem by induction over the number n of rows (and columns) of A.

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We will prove the theorem by induction over the number n of rows (and columns) of A.
If $n = 1$, then $\det A = a_{11}$,

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Proof of Theorem 2

We will prove the theorem by induction over the number n of rows (and columns) of A.
If $n = 1$, then $\det A = a_{11}$, so the theorem is true in this case.

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Proof of Theorem 2

We will prove the theorem by induction over the number n of rows (and columns) of A.
If $n = 1$, then $\det A = a_{11}$, so the theorem is true in this case.
Suppose $n > 1$ and that the theorem is true for $(n-1) \times (n-1)$ matrices.

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We will prove the theorem by induction over the number n of rows (and columns) of A.
If $n = 1$, then $\det A = a_{11}$, so the theorem is true in this case.
Suppose $n > 1$ and that the theorem is true for $(n-1) \times (n-1)$ matrices. Then

$$\det A = \begin{vmatrix} a_{11} & 0 \\ 0 & A_{11} \end{vmatrix} = a_{11} \det A_{11} = a_{11} a_{22} \dots a_{nn}.$$

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Proof of Theorem 2

We will prove the theorem by induction over the number n of rows (and columns) of A.

If $n = 1$, then $\det A = a_{11}$, so the theorem is true in this case.

Suppose $n > 1$ and that the theorem is true for $(n-1) \times (n-1)$ matrices. Then

$$\det A = \begin{vmatrix} a_{11} & 0 \\ 0 & A_{11} \end{vmatrix} = a_{11} \det A_{11} = a_{11} a_{22} \dots a_{nn}.$$

So it follows by induction that the theorem is true for all matrices A.

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Properties of determinants

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Properties of determinants

Theorem 3

Let A be a square matrix.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

Proof of Theorem 3

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Proof of Theorem 3

Let A be an $n \times n$ matrix and let E be an elementary $n \times n$ matrix.

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Proof of Theorem 3

Let A be an $n \times n$ matrix and let E be an elementary $n \times n$ matrix. We will show that

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement matrix,} \\ -1 & \text{if } E \text{ is a row interchange matrix,} \\ k & \text{if } E \text{ is a scale a row by } k \text{ matrix,} \end{cases}$$

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and that $\det(EA) = \det E \det A$.

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and that $\det(EA) = \det E \det A$.
We will prove this by induction over n.
If $n = 1$, then $E = [k]$ for some number k,

Proof of Theorem 3

Let A be an $n \times n$ matrix and let E be an elementary $n \times n$ matrix. We will show that

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement matrix,} \\ -1 & \text{if } E \text{ is a row interchange matrix,} \\ k & \text{if } E \text{ is a scale a row by } k \text{ matrix,} \end{cases}$$

and that $\det(EA) = \det E \det A$.
We will prove this by induction over n.
If $n = 1$, then $E = [k]$ for some number k, and then $\det(E) = k$ and $\det(EA) = \det(kA) = k \det A$.

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We will prove this by induction over n.
If $n = 1$, then $E = [k]$ for some number k, and then $\det(E) = k$ and $\det(EA) = \det(kA) = k \det A$. So the statement is true for $n = 1$.

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Proof of Theorem 3 (cont.)

Suppose $n = 2$.

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Proof of Theorem 3 (cont.)

Suppose $n = 2$. If $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

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Proof of Theorem 3 (cont.)

Suppose $n = 2$. If $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det E = 1$ and

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Proof of Theorem 3 (cont.)

Suppose $n = 2$. If $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det E = 1$ and

$$\det(EA) = \det \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix} = (a + kc)d - c(b + kd)$$

$$= ad + kcd - cd - ckd = ad - cd = \det(A).$$

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Proof of Theorem 3 (cont.)

Suppose $n = 2$. If $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det E = 1$ and

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One can in a similarly way prove that if $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then

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Proof of Theorem 3 (cont.)

Suppose $n = 2$. If $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det E = 1$ and

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$$= ad + kcd - cd - ckd = ad - cd = \det(A).$$

One can in a similarly way prove that if $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then $\det E = 1$ and $\det(EA) = \det A$, that if $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\det E = -1$ and $\det(EA) = -\det A$,

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Proof of Theorem 3 (cont.)

and that if $E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, then $\det E = k$ and $\det(EA) = k \det A$,

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Proof of Theorem 3 (cont.)

and that if $E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, then $\det E = k$ and $\det(EA) = k \det A$, so the statement is true for $n = 2$.

Proof of Theorem 3 (cont.)

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Proof of Theorem 3 (cont.)

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Proof of Theorem 3 (cont.)

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Proof of Theorem 3 (cont.)

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Proof of Theorem 3 (cont.)

and that if $E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, then $\det E = k$ and $\det(EA) = k \det A$, so the statement is true for $n = 2$. Suppose that $k > 2$ and that the statement holds for $n = k - 1$. Let E be an elementary $k \times k$ matrix. Choose i such that the ith row of E is equal to the ith row of I_k. Then E_{ii} is an elementary $(k-1) \times (k-1)$ matrix of the same kind as E, and $\det E = (-1)^{i+i} \det E_{ii} = \det E_{ii}$.

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Proof of Theorem 3 (cont.)

If A is a $k \times k$,

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Proof of Theorem 3 (cont.)

If A is a $k \times k$, then

$$\det(EA)$$

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Proof of Theorem 3 (cont.)

If A is a $k \times k$, then

$$\det(EA) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(EA)_{ij}$$

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Proof of Theorem 3 (cont.)

If A is a $k \times k$, then

$$\det(EA) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(EA)_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(E_{ii} A_{ij})$$

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Proof of Theorem 3 (cont.)

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$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det E_{ii} \det A_{ij})$$

$$= \det E \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

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Proof of Theorem 3 (cont.)

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$$= \det E \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \det E \det A.$$

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Proof of Theorem 3 (cont.)

If A is a $k \times k$, then

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It follows by induction over n that the statement, and thus the theorem, holds for all n.

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Properties of determinants

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Properties of determinants

- Suppose an $n \times n$ matrix A has been reduced to an echelon form U by row replacements and row interchanges.

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Properties of determinants

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- If there are r interchanges, then $\det(A) = (-1)^r \det(U)$.

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Properties of determinants

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- If there are r interchanges, then $\det(A) = (-1)^r \det(U)$.
- Since U is in echelon form, it is triangular, so $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.

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- If A is invertible, the entries $u_{11}, u_{22}, \ldots, u_{nn}$ are all pivots. Otherwise, at least one u_{ii} is zero.

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- Since U is in echelon form, it is triangular, so $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.
- If A is invertible, the entries $u_{11}, u_{22}, \ldots, u_{nn}$ are all pivots. Otherwise, at least one u_{ii} is zero.
- Thus,

$$\det(A) = \begin{cases} (-1)^r u_{11} u_{22} \ldots u_{nn} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

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Properties of determinants (cont.)

Thus we have proved:

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

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Proof of Theorem 5

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We will prove the theorem by induction over n where n is the number of rows of A.

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If $n = 1$, then $A^T = A$ from which it follows that $\det(A) = \det(A^T)$.

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If $n = 1$, then $A^T = A$ from which it follows that $\det(A) = \det(A^T)$.
Let k be a positive integer and assume that the theorem is true for all $k \times k$ matrices.

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Proof of Theorem 5

We will prove the theorem by induction over n where n is the number of rows of A.
If $n = 1$, then $A^T = A$ from which it follows that $\det(A) = \det(A^T)$.
Let k be a positive integer and assume that the theorem is true for all $k \times k$ matrices. Let $n = k + 1$ and let A be an $n \times n$ matrix.

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Proof of Theorem 5 (cont.)

Then

$\det(A)$

Proof of Theorem 5 (cont.)

Then

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1})$$

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Proof of Theorem 5 (cont.)

Then

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1}) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A_{i1})^T)$$

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Proof of Theorem 5 (cont.)

Then

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1}) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A_{i1})^T)$$

$$= \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A^T)_{1i})$$

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Proof of Theorem 5 (cont.)

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$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1}) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A_{i1})^T)$$

$$= \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A^T)_{1i})$$

$$= \sum_{i=1}^{n} (-1)^{1+i} (a^T)_{1i} \det((A^T)_{1i})$$

Proof of Theorem 5 (cont.)

Then

$$\begin{aligned}
\det(A) &= \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1}) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A_{i1})^T) \\
&= \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A^T)_{1i}) \\
&= \sum_{i=1}^{n} (-1)^{1+i} (a^T)_{1i} \det((A^T)_{1i}) = \det(A^T).
\end{aligned}$$

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Proof of Theorem 5 (cont.)

Then

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(A_{i1}) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A_{i1})^T)$$

$$= \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det((A^T)_{1i})$$

$$= \sum_{i=1}^{n} (-1)^{1+i} (a^T)_{1i} \det((A^T)_{1i}) = \det(A^T).$$

It follows by induction that $\det(A) = \det(A^T)$ for all square matrices.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 10 & 1 & -7 \\ 3 & 2 & -3 \\ -5 & 0 & 5 \end{bmatrix}$.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 10 & 1 & -7 \\ 3 & 2 & -3 \\ -5 & 0 & 5 \end{bmatrix}$.

It follows from Theorem 5 and Theorem 3 that if we add the third column to the first row, then that does not change the determinant.

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Example

Let us compute the determinant of $A = \begin{bmatrix} 10 & 1 & -7 \\ 3 & 2 & -3 \\ -5 & 0 & 5 \end{bmatrix}$.

It follows from Theorem 5 and Theorem 3 that if we add the third column to the first row, then that does not change the determinant. So

$$\det(A) = \begin{vmatrix} 10 & 1 & -7 \\ 3 & 2 & -3 \\ -5 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -7 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot 3 \cdot 5 = 30.$$

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Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

It follows from the theorem that if A and B are $n \times n$ matrices, then $\det(AB) = det(A)\det(B) = \det(B)\det(A) = \det(BA)$, even if $AB \neq BA$.

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Proof of Theorem 6

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If A is not invertible, then neither is AB,

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If A is not invertible, then neither is AB, so $\det(A)\det(B) = 0 = \det(AB)$ in that case.

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Proof of Theorem 6

If A is not invertible, then neither is AB, so
$\det(A)\det(B) = 0 = \det(AB)$ in that case.
If A is invertible, then A is row equivalent to I_n,

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Proof of Theorem 6

If A is not invertible, then neither is AB, so $\det(A)\det(B) = 0 = \det(AB)$ in that case.
If A is invertible, then A is row equivalent to I_n, so there are elementary matrices $E_1, E_2, \ldots, E_{p-1}, E_p$ such that
$A = E_p E_{p-1} \ldots E_2 E_1 I_n = E_p E_{p-1} \ldots E_2 E_1$,

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Proof of Theorem 6

If A is not invertible, then neither is AB, so $\det(A)\det(B) = 0 = \det(AB)$ in that case.
If A is invertible, then A is row equivalent to I_n, so there are elementary matrices $E_1, E_2, \ldots, E_{p-1}, E_p$ such that $A = E_pE_{p-1}\ldots E_2E_1I_n = E_pE_{p-1}\ldots E_2E_1$, and then

$$\begin{aligned}
\det(AB) &= \det(E_pE_{p-1}\ldots E_2E_1B) \\
&= \det(E_p)\det(E_{p-1}\ldots E_2E_1B) \\
&= \cdots = \det(E_p)\det(E_{p-1})\ldots\det(E_2)\det(E_1)\det(B) \\
&= \det(E_pE_{p-1}\ldots E_2E_1)\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

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Warnings

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A + B) \neq \det(A) + \det(B)$.

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A + B) \neq \det(A) + \det(B)$.
- In general, $\det(kA) \neq k \det(A)$.

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A+B) \neq \det(A) + \det(B)$.
- In general, $\det(kA) \neq k\det(A)$.

In fact, $\det(kA) = k^n \det(A)$.

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Problem 4 from June 2005

Find the determinant of the matrix $A = \begin{bmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

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Solution

Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

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Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

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Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix}$$

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Solution

$$\det A = \begin{vmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{vmatrix} = - \begin{vmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} = 8.$$

Problem 6 from August 2010

For which values of the parameter a are the vectors $\mathbf{v}_1 = (1, -3, a)$, $\mathbf{v}_2 = (0, 1, a)$ and $\mathbf{v}_3 = (a, 2, 0)$ linearly dependent?

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Solution

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Solution

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the matrix
$A = \begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$ is not invertible.

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Solution

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the matrix $A = \begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$ is not invertible.

$$\det A = \begin{vmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{vmatrix}$$

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Solution

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the matrix

$A = \begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$ is not invertible.

$$\det A = \begin{vmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ a & 0 \end{vmatrix} + a \begin{vmatrix} -3 & 1 \\ a & a \end{vmatrix}$$

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Solution

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the matrix
$A = \begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$ is not invertible.

$$\det A = \begin{vmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ a & 0 \end{vmatrix} + a \begin{vmatrix} -3 & 1 \\ a & a \end{vmatrix} = -2a - 4a^2$$

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Solution

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if the matrix
$A = \begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$ is not invertible.

$$\det A = \begin{vmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ a & 0 \end{vmatrix} + a \begin{vmatrix} -3 & 1 \\ a & a \end{vmatrix} = -2a - 4a^2$$
$$= -2a(1 + 2a).$$

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So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if $a = 0$ or $a = -1/2$.

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Problem 6 from June 2012

Let A be a 4×4 matrix. Let $B = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Assume that $\det(AB) = 4$. What is $\det(A)$?

Show that the equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has only the solution $x_1 = x_2 = x_3 = x_4 = 0$.

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Solution

Solution

$\det B$

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Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

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Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

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Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

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Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -3 + 1$$

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Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -3 + 1 = -2.$$

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Solution

$$\det B = \begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -3 + 1 = -2.$$

It follows that $\det A = \frac{\det(AB)}{\det B} = \frac{4}{-2} = -2$.

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Solution (cont.)

Since $\det A \neq 0$, A is invertible,

Solution (cont.)

Since $\det A \neq 0$, A is invertible, so the equation

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ has only the solution } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Tomorrow's lecture

Tomorrow we shall

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Tomorrow's lecture

Tomorrow we shall

- look at *Cramer's rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

Section 3.3 in "Linear Algebras and Its Applications" (pages 177–187).

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