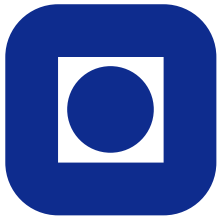

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TMA4115 - Calculus 3

Lecture 14, Feb 28

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday's lecture

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Yesterday's lecture

Yesterday we

- saw how we can *add* and *multiply* matrices,
- defined and studied *invertible* matrices.

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Today's lecture

Today's lecture

Today we shall

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Today's lecture

Today we shall

- further study *invertible* matrices,

Today's lecture

Today we shall

- further study *invertible* matrices,
- look at *the invertible matrix theorem*.

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Matrix multiplication

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Matrix multiplication

Definition

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.
That is,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \ldots \ A\mathbf{b}_p].$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

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The inverse of a matrix

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The inverse of a matrix

1. An $n \times n$ matrix A is said to be *invertible* if there exists an $n \times n$ matrix C such that $AC = CA = I_n$.

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1. An $n \times n$ matrix A is said to be *invertible* if there exists an $n \times n$ matrix C such that $AC = CA = I_n$.
2. There can at most be one $n \times n$ matrix C such that $AC = CA = I_n$. If such a C exists, then it is called the *inverse* of A and is denoted by A^{-1}.

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3. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix of T, then T is invertible if and only if A is.

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4. If T and A are invertible, then A^{-1} is the standard matrix of T^{-1}.

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3. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix of T, then T is invertible if and only if A is.
4. If T and A are invertible, then A^{-1} is the standard matrix of T^{-1}.
5. A matrix that is not invertible is sometimes called a *singular* matrix, and an invertible matrix is called a *nonsingular* matrix.

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Examples

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

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Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

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If A and C are not square matrices, then it can happen that

$AC = I$, but $CA \neq I$.

Examples

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.
Then $AC = CA = I_2$. So A is invertible and $A^{-1} = C$.
If A and C are not square matrices, then it can happen that $AC = I$, but $CA \neq I$. If for example $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ and

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

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Examples

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.
Then $AC = CA = I_2$. So A is invertible and $A^{-1} = C$.
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Examples

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.
Then $AC = CA = I_2$. So A is invertible and $A^{-1} = C$.
If A and C are not square matrices, then it can happen that $AC = I$, but $CA \neq I$. If for example $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AC = I_2$ and $CA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.
This reflects the fact that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be onto without being one-to-one if $n \neq m$.

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Using the inverse of a matrix to solve matrix equations

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Using the inverse of a matrix to solve matrix equations

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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Proof

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Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$.

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Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$.
Let $x = A^{-1}\mathbf{b}$.

Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Let $x = A^{-1}\mathbf{b}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b},$$

Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Let $x = A^{-1}\mathbf{b}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b},$$

so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

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Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$.
Let $x = A^{-1}\mathbf{b}$. Then

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so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.
Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$.
Let $x = A^{-1}\mathbf{b}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b},$$

so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.
Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{y} = I_n\mathbf{y} = (A^{-1}A)\mathbf{y} = A^{-1}(A\mathbf{y}) = A^{-1}\mathbf{b}.$$

Proof

Assume that A is invertible and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$.
Let $x = A^{-1}\mathbf{b}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b},$$

so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.
Assume that $\mathbf{x} = \mathbf{y}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{y} = I_n\mathbf{y} = (A^{-1}A)\mathbf{y} = A^{-1}(A\mathbf{y}) = A^{-1}\mathbf{b}.$$

So $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution to the equation $A\mathbf{x} = \mathbf{b}$.

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Example

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Example

Let us solve the linear system

$$\begin{aligned} 2x_1 + 5x_2 &= 3 \\ -3x_1 - 7x_2 &= 2 \end{aligned}$$

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Example

Let us solve the linear system

$$\begin{aligned} 2x_1 + 5x_2 &= 3 \\ -3x_1 - 7x_2 &= 2 \end{aligned}$$

We know that $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

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Example

Let us solve the linear system

$$\begin{aligned} 2x_1 + 5x_2 &= 3 \\ -3x_1 - 7x_2 &= 2 \end{aligned}$$

We know that $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. So

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -31 \\ 13 \end{bmatrix}$$

is the unique solution of the system.

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The inverse of an invertible 2×2 matrix

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The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

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The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

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The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The number $ad - bc$ is called the *determinant* of A.

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Proof

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Proof

Assume that $ad - bc \neq 0$ and let $C = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

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Assume that $ad - bc \neq 0$ and let $C = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then

$$AC = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad + b(-c) & a(-b) + ba \\ cd + d(-c) & c(-b) + da \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Proof

Assume that $ad - bc \neq 0$ and let $C = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then

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and

$$CA = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} da + (-b)c & db + (-b)d \\ (-c)a + ac & (-c)b + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Proof (cont.)

So A is invertible and $A^{-1} = C = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof (cont.)

So A is invertible and $A^{-1} = C = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Conversely, assume that $ad - bc = 0$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$.

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Proof (cont.)

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Proof (cont.)

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If $a \neq 0$ or $b \neq 0$, then $\begin{bmatrix} b \\ -a \end{bmatrix} \neq \mathbf{0}$, and $T\left(\begin{bmatrix} b \\ -a \end{bmatrix}\right) = A\begin{bmatrix} b \\ -a \end{bmatrix}$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} ab + b(-a) \\ cb + d(-a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

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Proof (cont.)

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Conversely, assume that $ad - bc = 0$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. If $a = b = 0$, then the columns of A do not span $\mathbb{R}^2$, so T is not onto, and hence not invertible, from which it follows that A is not invertible either.

If $a \neq 0$ or $b \neq 0$, then $\begin{bmatrix} b \\ -a \end{bmatrix} \neq \mathbf{0}$, and $T\left(\begin{bmatrix} b \\ -a \end{bmatrix}\right) = A\begin{bmatrix} b \\ -a \end{bmatrix}$ $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} ab + b(-a) \\ cb + d(-a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so T is not one-to-one, and hence not invertible, from which it follows that A is not invertible either.

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Example

Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. Let us show that A is invertible and find A^{-1}.

Solution

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Solution

$$3 \cdot 6 - 4 \cdot 5 = 18 - 20 = -2 \neq 0,$$

Solution

$3 \cdot 6 - 4 \cdot 5 = 18 - 20 = -2 \neq 0$, so A is invertible,

Solution

$3 \cdot 6 - 4 \cdot 5 = 18 - 20 = -2 \neq 0$, so A is invertible, and

$$A^{-1} = \frac{1}{3 \cdot 6 - 4 \cdot 5} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}.$$

Properties of invertible matrices

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Properties of invertible matrices

Theorem 6

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

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2. If A and B are invertible $n \times n$ matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

Properties of invertible matrices

Theorem 6

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2. If A and B are invertible $n \times n$ matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

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Proof

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Proof

1. Assume that A is invertible, and let $C = A$.

Proof

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.

Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$.

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$,

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$, and $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$,

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$, and $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$, so AB is invertible, and $(AB)^{-1} = C = B^{-1}A^{-1}$.

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$, and $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$, so AB is invertible, and $(AB)^{-1} = C = B^{-1}A^{-1}$.
3. Assume that A is invertible, and let $C = (A^{-1})^T$.

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$, and $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$, so AB is invertible, and $(AB)^{-1} = C = B^{-1}A^{-1}$.
3. Assume that A is invertible, and let $C = (A^{-1})^T$. Then $A^TC = A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$, and $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$, so AB is invertible, and $(AB)^{-1} = C = B^{-1}A^{-1}$.
3. Assume that A is invertible, and let $C = (A^{-1})^T$. Then $A^TC = A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$ and $CA^T = (A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n$,

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Proof

1. Assume that A is invertible, and let $C = A$. Then $A^{-1}C = A^{-1}A = I_n$ and $CA^{-1} = AA^{-1} = I_n$, so A^{-1} is invertible, and $(A^{-1})^{-1} = C = A$.
2. Assume that A and B are invertible, and let $C = B^{-1}A^{-1}$. Then $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$, and $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$, so AB is invertible, and $(AB)^{-1} = C = B^{-1}A^{-1}$.
3. Assume that A is invertible, and let $C = (A^{-1})^T$. Then $A^TC = A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$ and $CA^T = (A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n$, so A^T is invertible, and $(A^T)^{-1} = C = (A^{-1})^T$.

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Elementary matrices

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Elementary matrices

- An *elementary matrix* is one that is obtained by performing a single elementary row operation on an identity matrix.

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- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

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Elementary matrices

- An *elementary matrix* is one that is obtained by performing a single elementary row operation on an identity matrix.
- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.
- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

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Example

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Example

The matrices $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are elementary matrices.

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Example

The matrices $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are elementary matrices.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

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Example

The matrices $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are elementary matrices.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Then

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Example (cont.)

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{bmatrix}$$

Example (cont.)

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$$

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Example (cont.)

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$$

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Example (cont.)

Finally we see that $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

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Example (cont.)

Finally we see that $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$E_2^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

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Example (cont.)

Finally we see that $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$E_2^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and

$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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Invertible matrices

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Invertible matrices

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

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Proof

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Proof

Assume that A is invertible.

Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

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Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row.

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Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row. Since A is a square matrix, it follows that the reduced echelon form of A is I_n.

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Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row. Since A is a square matrix, it follows that the reduced echelon form of A is I_n. Thus, A is row equivalent to I_n.

Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row. Since A is a square matrix, it follows that the reduced echelon form of A is I_n. Thus, A is row equivalent to I_n. Conversely, assume that A is row equivalent to I_n.

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Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row. Since A is a square matrix, it follows that the reduced echelon form of A is I_n. Thus, A is row equivalent to I_n. Conversely, assume that A is row equivalent to I_n. Then there is a sequence of elementary row operations that transforms A to I_n.

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Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row. Since A is a square matrix, it follows that the reduced echelon form of A is I_n. Thus, A is row equivalent to I_n. Conversely, assume that A is row equivalent to I_n. Then there is a sequence of elementary row operations that transforms A to I_n. Since each elementary row operation corresponds to left multiplication by an elementary matrix, it follows that there are elementary matrices $E_1, E_2, \ldots, E_p$ such that $E_p \ldots E_2 E_1 A = I_n$.

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Proof

Assume that A is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. It follows that the columns of A span $\mathbb{R}^n$, and thus that A has a pivot position in every row. Since A is a square matrix, it follows that the reduced echelon form of A is I_n. Thus, A is row equivalent to I_n. Conversely, assume that A is row equivalent to I_n. Then there is a sequence of elementary row operations that transforms A to I_n. Since each elementary row operation corresponds to left multiplication by an elementary matrix, it follows that there are elementary matrices $E_1, E_2, \ldots, E_p$ such that $E_p \ldots E_2 E_1 A = I_n$. Let $C = E_p \ldots E_2 E_1$.

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Proof (cont.)

Since each elementary matrix is invertible, it follows that C is invertible and that $C^{-1} = E_1^{-1} E_2^{-1} \dots E_p^{-1}$.

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Proof (cont.)

Since each elementary matrix is invertible, it follows that C is invertible and that $C^{-1} = E_1^{-1} E_2^{-1} \dots E_p^{-1}$.

$$\begin{aligned} A = IA = (C^{-1}C)A = C^{-1}(CA) = C^{-1}(E_p \dots E_2 E_1 A) \\ = C^{-1} I_n = C^{-1} \end{aligned}$$

Proof (cont.)

Since each elementary matrix is invertible, it follows that C is invertible and that $C^{-1} = E_1^{-1} E_2^{-1} \dots E_p^{-1}$.

$$\begin{aligned} A = IA = (C^{-1}C)A = C^{-1}(CA) = C^{-1}(E_p \dots E_2 E_1 A) \\ = C^{-1} I_n = C^{-1} \end{aligned}$$

so A is invertible and $A^{-1} = C = E_p \dots E_2 E_1 I_n$.

Proof (cont.)

Since each elementary matrix is invertible, it follows that C is invertible and that $C^{-1} = E_1^{-1} E_2^{-1} \dots E_p^{-1}$.

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so A is invertible and $A^{-1} = C = E_p \dots E_2 E_1 I_n$. Thus, the elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

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An algorithm for finding A^{-1}

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An algorithm for finding A^{-1}

1. Row reduce the augmented matrix $[A\ I]$.

An algorithm for finding A^{-1}

1. Row reduce the augmented matrix $[A\ I]$.
2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.

An algorithm for finding A^{-1}

1. Row reduce the augmented matrix $[A\ I]$.
2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.
3. If A is not row equivalent to I, then A is not invertible.

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Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.

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Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$. Let us show that A is invertible and find A^{-1}.

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Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$. Let us show that A is invertible and find A^{-1}. We reduce the matrix $[A\ I]$ to its reduced echelon form.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$. Let us show that A is invertible and find A^{-1}. We reduce the matrix $[A\ I]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

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Example (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

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Example (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

It follows that A is invertible, and that

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}.$$

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The invertible matrix theorem

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The invertible matrix theorem

Theorem 8

Let A be a square $n \times n$ matrix.

The invertible matrix theorem

Theorem 8

Let A be a square $n \times n$ matrix. Then the following statements are equivalent, that is, for a given A, the following statements are either all true or all false.

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The invertible matrix theorem

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The invertible matrix theorem

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Let A be a square $n \times n$ matrix. Then the following statements are equivalent, that is, for a given A, the following statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I_n.

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The invertible matrix theorem

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1. A is invertible.
2. A is row equivalent to I_n.
3. A has n pivot positions.

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The invertible matrix theorem

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1. A is invertible.
2. A is row equivalent to I_n.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

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The invertible matrix theorem

Theorem 8

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1. A is invertible.
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5. The columns of A form a linearly independent set.

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The invertible matrix theorem

Theorem 8

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1. A is invertible.
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3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

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7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

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7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

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10. There is an $n \times n$ matrix C such that $CA = I_n$.

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11. There is an $n \times n$ matrix D such that $AD = I_n$.

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7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
10. There is an $n \times n$ matrix C such that $CA = I_n$.
11. There is an $n \times n$ matrix D such that $AD = I_n$.
12. A^T is invertible.

Proof

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Proof

1 $\iff$ 2: We have already seen that A is invertible if and only if it is row equivalent to I_n.

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Proof

1 $\iff$ 2: We have already seen that A is invertible if and only if it is row equivalent to I_n.

2 $\iff$ 3: Since A is square matrix, it follows that A has n pivot positions if and only if the reduced echelon form of A is I_n.

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Proof

$1 \iff 2$: We have already seen that A is invertible if and only if it is row equivalent to I_n.

$2 \iff 3$: Since A is square matrix, it follows that A has n pivot positions if and only if the reduced echelon form of A is I_n.

$1 \iff 12$: We have already seen that A^T is invertible if A is, and since $A = (A^T)^T$, it follows that A is invertible if A^T is.

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Proof

$1 \iff 2$: We have already seen that A is invertible if and only if it is row equivalent to I_n.

$2 \iff 3$: Since A is square matrix, it follows that A has n pivot positions if and only if the reduced echelon form of A is I_n.

$1 \iff 12$: We have already seen that A^T is invertible if A is, and since $A = (A^T)^T$, it follows that A is invertible if A^T is.

$1 \implies 10$: If A is invertible and we let $C = A^{-1}$, then $CA = A^{-1}A = I_n$.

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Proof

$1 \iff 2$: We have already seen that A is invertible if and only if it is row equivalent to I_n.

$2 \iff 3$: Since A is square matrix, it follows that A has n pivot positions if and only if the reduced echelon form of A is I_n.

$1 \iff 12$: We have already seen that A^T is invertible if A is, and since $A = (A^T)^T$, it follows that A is invertible if A^T is.

$1 \implies 10$: If A is invertible and we let $C = A^{-1}$, then $CA = A^{-1}A = I_n$.

$10 \implies 4$: If $CA = I_n$ and $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = I_n\mathbf{x} = CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$.

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Proof (cont.)

4 $\Longrightarrow$ 6: We have already seen that the linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one if (and only if) the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Proof (cont.)

4 $\Longrightarrow$ 6: We have already seen that the linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one if (and only if) the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

6 $\Longrightarrow$ 5: We have already seen that the columns of A form a linearly independent set if (and only if) the linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.

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Proof (cont.)

- 4 $\Longrightarrow$ 6: We have already seen that the linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one if (and only if) the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- 6 $\Longrightarrow$ 5: We have already seen that the columns of A form a linearly independent set if (and only if) the linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
- 5 $\Longrightarrow$ 3: If the columns of A form a linearly independent set, then A has a pivot position in every column, and thus n pivot positions.

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Proof (cont.)

1 $\Longrightarrow$ 11: If A is invertible and we let $D = A^{-1}$, then $AD = AA^{-1} = I_n$.

Proof (cont.)

1 $\Longrightarrow$ 11: If A is invertible and we let $D = A^{-1}$, then $AD = AA^{-1} = I_n$.

11 $\Longrightarrow$ 7: Assume that $AD = I_n$ and let $\mathbf{b} \in \mathbb{R}^n$. If we let $\mathbf{x} = D\mathbf{b}$, then $A\mathbf{x} = AD\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$, so the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

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Proof (cont.)

$1 \implies 11$: If A is invertible and we let $D = A^{-1}$, then $AD = AA^{-1} = I_n$.

$11 \implies 7$: Assume that $AD = I_n$ and let $\mathbf{b} \in \mathbb{R}^n$. If we let $\mathbf{x} = D\mathbf{b}$, then $A\mathbf{x} = AD\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$, so the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

$7 \implies 9$: If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then every $b \in \mathbb{R}^n$ is in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ which is therefore onto.

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Proof (cont.)

9 $\Longrightarrow$ 8: We have already seen that the columns of A span $\mathbb{R}^n$ if (and only if) the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

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Proof (cont.)

9 $\Longrightarrow$ 8: We have already seen that the columns of A span $\mathbb{R}^n$ if (and only if) the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

8 $\Longrightarrow$ 3: If the columns of A span $\mathbb{R}^n$, then A has a pivot position in every row, and thus n pivot positions.

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Examples

Let us use the invertible matrix theorem to determine which of the following matrices are invertible.

$$\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 8 \\ -3 & -2 & 5 \end{bmatrix}$$

Solution

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Solution

Every linear combination of the columns of $\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ has

a 0 in the second row,

Solution

Every linear combination of the columns of $\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ has a 0 in the second row, so the columns of $\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ do not span $\mathbb{R}^3$,

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Solution

Every linear combination of the columns of $\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ has a 0 in the second row, so the columns of $\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ do not span $\mathbb{R}^3$, and $\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ is therefore not invertible.

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Solution (cont.)

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Solution (cont.)

The columns of $\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$ are linear independent,

Solution (cont.)

The columns of $\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$ are linear independent, so $\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$ is invertible.

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Solution (cont.)

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Solution (cont.)

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 8 \\ -3 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 8 \\ -3 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 8 \\ -3 & -2 & 5 \end{bmatrix}$ does not have a pivot position in every row and is therefore not invertible.

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Problem 4 from the exam from December 2010

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Show that A is invertible and find A^{-1}.

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Solution

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Solution

We reduce the matrix $[A\ I]$ to its reduced echelon form.

Solution

We reduce the matrix $[A\ I]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

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Solution (cont.)

It follows that A is invertible, and that $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$.

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Problem 7 from the exam from August 2011

Let A and B be square $n \times n$ matrices and assume that AB is invertible. Show that A is also invertible.

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Solution

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Solution

Let $D = B(AB)^{-1}$.

Solution

Let $D = B(AB)^{-1}$. Then
$AD = A(B(AB)^{-1}) = (AB)(AB)^{-1} = I_n$,

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Solution

Let $D = B(AB)^{-1}$. Then
$AD = A(B(AB)^{-1}) = (AB)(AB)^{-1} = I_n$, so it follows from the invertible matrix theorem that A is invertible.

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Example

Let E and F be $n \times n$ matrices and assume that $EF = I_n$. Let us show that E and F commute.

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Solution

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Solution

It follows from the invertible matrix theorem that E is invertible (because $EF = I_n$).

Solution

It follows from the invertible matrix theorem that E is invertible (because $EF = I_n$).

$$F = IF = (E^{-1}E)F = E^{-1}(EF) = E^{-1}I_n = E^{-1}$$

Solution

It follows from the invertible matrix theorem that E is invertible (because $EF = I_n$).

$$F = IF = (E^{-1}E)F = E^{-1}(EF) = E^{-1}I_n = E^{-1}$$

so $FE = E^{-1}E = I_n = EF$ which shows that E and F commute.

Plan for next week

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Plan for next week

Wednesday we shall introduce and study *determinants*
Sections 3.1–3.2 in “Linear Algebras and Its Applications” (pages 163–177).

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Plan for next week

Wednesday we shall introduce and study *determinants*
Sections 3.1–3.2 in “Linear Algebras and Its Applications” (pages 163–177).

Thursday we shall

- look at *Cramer’s rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

Section 3.3 in “Linear Algebras and Its Applications” (pages 177–187).

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