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# TMA4115 - Calculus 3

## Lecture 12, Feb 21

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Norwegian University of Science and Technology

Spring 2013

# Yesterday's lecture

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Yesterday we introduced and studied

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- *linear transformations*,
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- *onto* linear transformations,
- *one-to-one* linear transformations.

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# Today's lecture

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- look at *applications* of linear models,

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# Today's lecture

Today we shall

- look at *applications* of linear models,
- look at *Maple* and *WolframAlpha*.

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# Linear transformations

# Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars $c$ and all $\mathbf{u}$ in $\mathbb{R}^n$.

# The matrix of a linear transformation

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# The matrix of a linear transformation

**Theorem 10**

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $A$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In fact, if we for each $j = 1, \dots n$ let $\mathbf{e}_j$ be the $j$th column of the identity matrix $I_n$, then $A$ is the $m \times n$ matrix $[T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_n)]$ whose $j$th column is the vector $T(\mathbf{e}_j)$.

The matrix $A$ is called the *standard matrix* of $T$.

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# One-to-one transformations

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# One-to-one transformations

**Definition of one-to-one transformations**

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* (or *injective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that each $\mathbf{b}$ in $\mathbb{R}^m$ does not have to be in the image of $T$ in order for $T$ to be one-to-one.

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## Theorem 11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

# Onto transformations

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# Onto transformations

**Definition of onto transformations**

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* (or *surjective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that a transformation $T$ is onto if and only if the image of $T$ is all of $\mathbb{R}^m$.

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# Onto and one-to-one linear transformations

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# Onto and one-to-one linear transformations

**Theorem 12**

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:

1. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$.
2. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.

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# Example

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# Example

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

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It is easy to check that $T$ is linear and that the standard matrix of $T$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

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It is easy to check that $T$ is linear and that the standard matrix of $T$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Since the columns of the standard matrix of $T$ are linearly dependent and span $\mathbb{R}^2$, it follows that $T$ is onto, but not one-to-one.

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# Example

# Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city. In 2000, there were 600,000 residents in the city and 400,000 in the suburbs. Let us fin the population in the city in the suburbs for the years 2001 and 2002.

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# Solution

# Solution

For $i = 0, 1, 2$, let $r_i$ be the population in the city in year $2000 + i$, and let $s_i$ be the population in the suburbs in year $2000 + i$.

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# Solution

For $i = 0, 1, 2$, let $r_i$ be the population in the city in year $2000 + i$, and let $s_i$ be the population in the suburbs in year $2000 + i$. Then

$$r_{i+1} = 0.95r_i + 0.03s_i$$
$$s_{i+1} = 0.005r_i + 0.97s_i$$

# Solution

For $i = 0, 1, 2$, let $r_i$ be the population in the city in year $2000 + i$, and let $s_i$ be the population in the suburbs in year $2000 + i$. Then

$$r_{i+1} = 0.95r_i + 0.03s_i$$
$$s_{i+1} = 0.005r_i + 0.97s_i$$

So if we let $\mathbf{x}_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}$, then $\mathbf{x}_{i+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \mathbf{x}_i$.

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# Solution (cont.)

$$\mathbf{x}_0 = \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix}$$

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# Solution (cont.)

$$\mathbf{x}_0 = \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} \text{ so}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} = \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix}$$

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$$\mathbf{x}_2 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix} = \begin{bmatrix} 565,440 \\ 434,560 \end{bmatrix}.$$

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# Example

# Example

The amounts of protein, carbohydrate and fat in nonfat milk, soy flour and whey are given by the following table.

| Nutrient | Nonfat milk | Soy flour | Whey |
|---|---|---|---|
| Protein | 36 | 51 | 13 |
| Carbohydrate | 52 | 34 | 74 |
| Fat | 0 | 7 | 1.1 |

Table: Amounts (g) supplied per 100 g of ingredient

Let us find the amounts of nonfat milk, soy flour and whey that would give a product containing 33 g protein, 45 g carbohydrate and 3 g fat.

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# Solution

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# Solution

Let $x_1$, $x_2$ and $x_3$ denote the number of units (100 g) of nonfat milk, soy flour, and whey, respectively, in the product.

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Let $x_1$, $x_2$ and $x_3$ denote the number of units (100 g) of nonfat milk, soy flour, and whey, respectively, in the product. Then we have that

$$\begin{aligned} 36x_1 + 51x_2 + 13x_3 &= 33 \\ 52x_1 + 34x_2 + 74x_3 &= 45 \\ 7x_2 + 1.1x_3 &= 3 \end{aligned}$$

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# Solution (cont.)

We write down the augmented matrix of the system and reduce it to reduced echelon form.

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$$\begin{bmatrix} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 1.1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0.277 \\ 0 & 1 & 0 & 0.392 \\ 0 & 0 & 1 & 0.233 \end{bmatrix}$$

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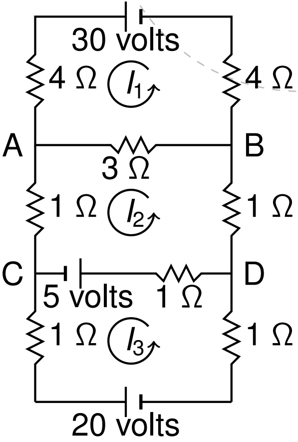
We see that $x_1 = 0.277$, $x_2 = 0.392$ and $x_3 = 0.233$, so the product must consists of 27.7 g nonfat milk, 39.2 g soy flour, and 23.3 g whey.

# Example

# Example

Kirchhoff's voltage law says that the directed sum of the electrical potential differences (voltage) around any closed network is zero. Ohm's law says that the electrical potential difference across a resistor is the product of the resistance and the current flow.
Let us use these laws to determine the current flows $I_1$, $I_2$, and $I_3$ in the following network.



# Solution

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# Solution

We have that

$$
\begin{aligned}
4I_1 + 3(I_1 - I_2) + 4I_1 - 30 &= 0 \\
I_2 + (I_2 - I_3) + I_2 + 3(I_2 - I_1) - 5 &= 0 \\
I_3 + 20 + I_3 + (I_3 - I_2) + 5 &= 0
\end{aligned}
$$

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# Solution

We have that

$$\begin{aligned} 4I_1 + 3(I_1 - I_2) + 4I_1 - 30 &= 0 \\ I_2 + (I_2 - I_3) + I_2 + 3(I_2 - I_1) - 5 &= 0 \\ I_3 + 20 + I_3 + (I_3 - I_2) + 5 &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} 11I_1 - 3I_2 &= 30 \\ -3I_1 + 6I_2 - I_3 &= 5 \\ -I_2 + 3I_3 &= -25 \end{aligned}$$

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$$\begin{bmatrix} 11 & -3 & 0 & 30 \\ -3 & 6 & -1 & 5 \\ 0 & -1 & 3 & -25 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

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It follows that $I_1 = 3$ amps, $I_2 = 1$ amps, and that $I_3 = -8$ amps.

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It follows that $I_1 = 3$ amps, $I_2 = 1$ amps, and that $I_3 = -8$ amps. That $I_3$ is negative means that current in loop 3 flows in the direction opposite to the one indicated in the figure.

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# Mathematical software

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1. Numerical analysis software
2. Computer algebra systems

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# Numerical analysis software

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Numerical analysis software is primarily used in applied mathematics. *MATLAB* is the main numerical analysis software.

# Computer algebra systems

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The main computer algebra systems are *Maple* and *Mathematica*.

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*WolframAlpha* is an online service that answers factual queries directly by computing the answer from structured data, rather than providing a list of documents or web pages that might contain the answer as a search engine might.

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# Plan for next week

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Wednesday we shall

- see how we can *add* and *multiply* matrices,
- define and study *invertible* matrices.

Sections 2.1–2.2 in “Linear Algebras and Its Applications” (pages 91–111).

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Thursday we shall

- further study *invertible* matrices,
- look at *the invertible matrix theorem*.

Section 2.3 in "Linear Algebras and Its Applications" (pages 111–116).