

#### TMA4115 - Calculus 3 Lecture 11, Feb 20

Toke Meier Carlsen Norwegian University of Science and Technology Spring 2013



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• introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,



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- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector* form,
- looked at applications of linear systems,
- introduced and studied *linear dependence* and *linear independence* of vectors.





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We shall introduce and study



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• linear transformations,



We shall introduce and study

- linear transformations,
- the standard matrix of a linear transformation,



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- the standard matrix of a linear transformation,
- onto linear transformations,



We shall introduce and study

- linear transformations,
- the standard matrix of a linear transformation,
- onto linear transformations,
- one-to-one linear transformations.





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A transformation (or function or mapping) T from ℝ<sup>n</sup> to ℝ<sup>m</sup> is a rule that assigns to each vector x in ℝ<sup>n</sup> a vector T(x) in ℝ<sup>m</sup>.



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- The set  $\mathbb{R}^n$  is called domain of T, and  $\mathbb{R}^m$  is called the codomain of T.



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- The notation *T* : ℝ<sup>n</sup> → ℝ<sup>m</sup> indicates that the domain of *T* is ℝ<sup>n</sup> and the codomain is ℝ<sup>m</sup>.



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- For x in ℝ<sup>n</sup>, the vector T(x) in ℝ<sup>m</sup> is called the image of x (under the action of T).
- The set of all images  $T(\mathbf{x})$  is called the range (or image) of T.



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For each 
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$
, let  
$$T\left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \cos(x_1 x_2 x_3 \pi) \end{bmatrix}$$



For each 
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$
, let
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Then *T* is a transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .



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Then *T* is a transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ . The image of  $\begin{bmatrix} 1\\1\\1 \end{bmatrix}$  is  $\begin{bmatrix} \sqrt{1^2 + 1^2 + 1^2}\\\cos(\pi) \end{bmatrix} = \begin{bmatrix} \sqrt{3}\\-1 \end{bmatrix}$ .



### Example (cont.)

The range of 
$$T$$
 is  $\left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : y \ge 0, y_2 \in [-1, 1] \right\}$  (this is not completely obvious).



# Example (cont.)

for all  $\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \in \mathbb{R}^3$ .

The range of *T* is  $\left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : y \ge 0, y_2 \in [-1, 1] \right\}$  (this is not completely obvious). If  $f_1(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$  and  $f_2(x_1, x_2, x_3) = \cos(x_1 x_2 x_3 \pi)$ , then  $f_1$  and  $f_2$  are real-valued functions on  $\mathbb{R}^3$  and

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}f_1(x_1,x_2,x_3)\\f_2(x_1,x_2,x_3)\end{bmatrix}$$



# Transformations and real-valued functions of several variables



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# Transformations and real-valued functions of several variables

If  $f_1, f_2, \ldots, f_m$  are real-valued functions on  $\mathbb{R}^n$ , then

$$T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}f_1(x_1,x_2,\ldots,x_n)\\f_2(x_1,x_2,\ldots,x_n)\\\vdots\\f_m(x_1,x_2,\ldots,x_n)\end{bmatrix}$$

defines a transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .



# Transformations and real-valued functions of several variables

Conversely, if *T* is a transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , then there exist real-valued functions  $f_1, f_2, \ldots, f_m$  on  $\mathbb{R}^n$  such that

$$T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}f_1(x_1, x_2, \dots, x_n)\\f_2(x_1, x_2, \dots, x_n)\\\vdots\\f_m(x_1, x_2, \dots, x_n)\end{bmatrix}$$



for all



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For each 
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$
, let
$$S\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \\ 3x_2 \end{bmatrix}$$



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For each 
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Then *S* is a transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ .



For each 
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$
, let $S\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \\ 3x_2 \end{bmatrix}$ 

Then *S* is a transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ . Notice that

$$S\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1\begin{bmatrix}2\\1\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\0\\3\end{bmatrix} = \begin{bmatrix}2 & -1\\1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

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# Example (cont.)

It follows that the range of *S* is Span  $\left\{ \begin{bmatrix} 2\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\3 \end{bmatrix} \right\}$ .



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#### **Matrix transformations**



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#### **Matrix transformations**

If *A* is an  $m \times n$  matrix and we for every **x** in  $\mathbb{R}^n$  let  $T(\mathbf{x}) = A\mathbf{x}$ , then *T* is a transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .



### **Matrix transformations**

If *A* is an  $m \times n$  matrix and we for every **x** in  $\mathbb{R}^n$  let  $T(\mathbf{x}) = A\mathbf{x}$ , then *T* is a transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . The range of *A* is Span $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  where  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  are the columns of *A*.





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A transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is *linear* if:



A transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is *linear* if:

•  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$ ;



A transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is *linear* if:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$ ;
- 2  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars c and all  $\mathbf{u}$  in  $\mathbb{R}^n$ .



#### Matrix transformations are linear



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### Matrix transformations are linear

Every matrix transformation is linear because if A is an  $m \times n$  matrix, then  $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$  for all  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$  and  $A(c\mathbf{u}) = cA\mathbf{u}$  for all scalars c and all  $\mathbf{u}$  in  $\mathbb{R}^n$ .





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If  $T : \mathbb{R}^n \to \mathbb{R}^m$  is a linear transformation, then:



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If  $T : \mathbb{R}^n \to \mathbb{R}^m$  is a linear transformation, then:

•  $T(\mathbf{0}) = \mathbf{0}$  because  $T(\mathbf{0}) = T(\mathbf{0}\mathbf{u}) = \mathbf{0}T(\mathbf{u}) = \mathbf{0}$  for any vector  $\mathbf{u}$  in  $\mathbb{R}^n$ .



If  $T : \mathbb{R}^n \to \mathbb{R}^m$  is a linear transformation, then:

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- 2  $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$  for all scalars c and d and all vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$ , because  $T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ .





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A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on material, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product C, the company spends \$0.40 on material, \$0.30 on labor, and \$0.15 on overhead.



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Let  $x_1$  denote the value of product B and  $x_2$  the value of product C that the company manufactures.



A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on material, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product C, the company spends \$0.40 on material, \$0.30 on labor, and \$0.15 on overhead.

Let  $x_1$  denote the value of product B and  $x_2$  the value of product C that the company manufactures.

Let  $y_1$ ,  $y_2$  and  $y_3$  denote the costs of material, labor and overhead the company spends on this.



Then

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.45x_1 + 0.40x_2 \\ 0.25x_1 + 0.30x_2 \\ 0.15x_1 + 0.15x_2 \end{bmatrix}$$



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Then

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.45x_1 + 0.40x_2 \\ 0.25x_1 + 0.30x_2 \\ 0.15x_1 + 0.15x_2 \end{bmatrix}$$

So if we define  $T : \mathbb{R}^2 \to \mathbb{R}^3$  by  $T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40\\ 0.25 & 0.30\\ 0.15 & 0.15 \end{bmatrix} \mathbf{x}$   $\begin{bmatrix} v_1 \end{bmatrix}$ 

then 
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = T(\mathbf{x}).$$





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# Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left( \begin{bmatrix} 1\\0 \end{bmatrix} \right) = \begin{bmatrix} 1\\3\\-1 \end{bmatrix}$ and $T\left( \begin{bmatrix} 0\\1 \end{bmatrix} \right) = \begin{bmatrix} -3\\5\\7 \end{bmatrix}$ .



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# Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\-1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\5\\7\end{bmatrix}$ . • Let $\mathbf{u} = \begin{bmatrix}2\\-1\end{bmatrix}$ . Let us find the image $T(\mathbf{u})$ of $\mathbf{u}$ under T.



Let 
$$T : \mathbb{R}^2 \to \mathbb{R}^3$$
 be a linear transformation such that  
 $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\-1\end{bmatrix}$  and  $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\5\\7\end{bmatrix}$ .  
• Let  $\mathbf{u} = \begin{bmatrix}2\\-1\end{bmatrix}$ . Let us find the image  $T(\mathbf{u})$  of  $\mathbf{u}$  under  $T$ .  
 $T(\mathbf{u}) = T\left(2\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$   
 $= 2\begin{bmatrix}1\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\5\\7\end{bmatrix} = \begin{bmatrix}5\\1\\-9\end{bmatrix}$   
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2 Let us find the image under T of  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .



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2 Let us find the image under *T* of  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .

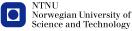
$$T(\mathbf{x}) = T\left(x_1 \begin{bmatrix} 1\\0 \end{bmatrix} + x_2 \begin{bmatrix} 0\\1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1\\0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0\\1 \end{bmatrix}\right)$$
$$= x_1 \begin{bmatrix} 1\\3\\-1 \end{bmatrix} + x_2 \begin{bmatrix} -3\\5\\7 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2\\3x_1 + 5x_2\\-x_1 + 7x_2 \end{bmatrix}$$



2 Let us find the image under T of  $\mathbf{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$ .

$$T(\mathbf{x}) = T\left(x_1 \begin{bmatrix} 1\\0 \end{bmatrix} + x_2 \begin{bmatrix} 0\\1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1\\0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0\\1 \end{bmatrix}\right)$$
$$= x_1 \begin{bmatrix} 1\\3\\-1 \end{bmatrix} + x_2 \begin{bmatrix} -3\\5\\7 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2\\3x_1 + 5x_2\\-x_1 + 7x_2 \end{bmatrix}$$

It follows that  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^2$  where  $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}.$ NTNU

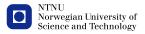


Solution Is there more than one **x** whose image under *T* is  $\begin{bmatrix} 5\\1\\-9 \end{bmatrix}$ ?



3 Is there more than one **x** whose image under *T* is  $\begin{bmatrix} 5\\1\\-9 \end{bmatrix}$ ?

The question is equivalent to the question: Does the equation  $A\mathbf{x} = \begin{bmatrix} 5\\1\\-9 \end{bmatrix}$  have more than one solution?



Solution Is there more than one **x** whose image under *T* is  $\begin{bmatrix} 5\\1\\-9 \end{bmatrix}$ ?

The question is equivalent to the question: Does the equation  $A\mathbf{x} = \begin{bmatrix} 5\\1\\-9 \end{bmatrix}$  have more than one solution?

To answer that question we reduce the augmented matrix of the equation to an echelon form.



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$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 5 & 1 \\ -1 & 7 & -9 \end{bmatrix}$$



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$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 5 & 1 \\ -1 & 7 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 14 & -14 \\ 0 & 4 & -4 \end{bmatrix}$$



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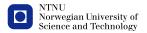
$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 5 & 1 \\ -1 & 7 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 14 & -14 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$



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$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 5 & 1 \\ -1 & 7 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 14 & -14 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that there is no free variable, so there is only one **x** such that  $T(\mathbf{x}) = \begin{bmatrix} 5\\1\\-9 \end{bmatrix}$ .



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# • Let $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ . Let us determine if $\mathbf{c}$ is in the range of T.



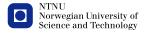
Let  $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ . Let us determine if  $\mathbf{c}$  is in the range of T.  $\mathbf{c}$  is in the range of T if and only if the equation  $A\mathbf{x} = \mathbf{c}$  is consistent.



• Let  $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ . Let us determine if  $\mathbf{c}$  is in the range of  $\mathcal{T}$ .

**c** is in the range of T if and only if the equation  $A\mathbf{x} = \mathbf{c}$  is consistent.

In order to determine whether the equation  $A\mathbf{x} = \mathbf{c}$  is consistent or not, we reduce the augmented matrix  $[A \mathbf{c}]$ to an echelon form.



$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix}$$



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$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix}$$



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$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$



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#### Example (cont.)

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$



#### Example (cont.)

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$



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#### Example (cont.)

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

We see that the system is inconsistent, so c is not in the range of T.





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#### Theorem 10

Let  $T : \mathbb{R}^n \to \mathbb{R}^m$  be a linear transformation. Then there exists a unique matrix A such that  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ .



#### Theorem 10

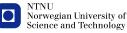
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The matrix A is called the standard matrix of T.





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Let  $A = [T(e_1) \dots T(e_n)].$ 



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## Let $A = [T(\mathbf{e}_1) \dots T(\mathbf{e}_n)]$ . We must show that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ .



Let  $A = [T(\mathbf{e}_1) \dots T(\mathbf{e}_n)]$ . We must show that  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ . Let  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  be a vector in  $\mathbb{R}^n$ .



## Proof (cont.)

#### Then

$$T(\mathbf{x}) = T\left( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = T\left( x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right)$$
$$= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n)$$
$$= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \dots + x_n T(\mathbf{e}_n)$$
$$= A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}.$$

#### Example



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#### Example

Let  $T : \mathbb{R}^3 \to \mathbb{R}^3$  be a linear transformation such that  $T(x_1, x_2, x_3) = (x_1 - 3x_2 - x_3, x_2 + 4x_3, 2x_1 + 9x_2 + 5x_3)$ . Let us find the standard matrix of T.





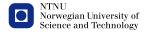
TMA4115 - Calculus 3, Lecture 11, Feb 20, page 27

$$T(\mathbf{e}_1) = T(1,0,0) = (1,0,2) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$



TMA4115 - Calculus 3, Lecture 11, Feb 20, page 27

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TMA4115 - Calculus 3, Lecture 11, Feb 20, page 27

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$$T(\mathbf{e}_{3}) = T(0,0,1) = (-1,4,5) = \begin{bmatrix} -1\\4\\5 \end{bmatrix},$$
 so the standard matrix of T is  $\begin{bmatrix} 1 & -3 & -1\\0 & 1 & 4\\2 & 9 & 5 \end{bmatrix}.$ 

#### Example



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#### Example

Let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be the function given by  $T(\mathbf{x}) = \mathbf{y}$  where  $\mathbf{y}$  is the vector we obtain from rotating  $\mathbf{x}$  by an angle of  $\theta$  around zero.

Let us show that T is a linear transformation and find its standard matrix.



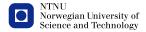


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Let 
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
 be a vector in  $\mathbb{R}^2$ , and let  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\mathbf{x})$ .



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 $x_1 = r \cos \phi$  and  $x_2 = r \sin \phi$ , so  $y_1 = r \cos(\phi + \theta) =$   
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So if we let  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , then  $T(\mathbf{x}) = A\mathbf{x}$  for every  $\mathbf{x}$   
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So if we let  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , then  $T(\mathbf{x}) = A\mathbf{x}$  for every  $\mathbf{x}$   
in  $\mathbb{R}^2$ .  
It follows that  $T$  is linear, and that  $A$  is the standard matrix of  
 $T$ .





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Definition of one-to-one transformations

A transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is said to be *one-to-one* (or *injective*) if each **b** in  $\mathbb{R}^m$  is the image of at most one **x** in  $\mathbb{R}^n$ .



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Notice that each **b** in  $\mathbb{R}^m$  does not have to be in the image of T in order for T to be one-to-one.



#### Theorem 11

A linear transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is one-to-one if and only the equation  $T(\mathbf{x}) = \mathbf{0}$  has only the trivial solution.





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We have that  $T(\mathbf{0}) = \mathbf{0}$  since T is linear.



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We have that  $T(\mathbf{0}) = \mathbf{0}$  since *T* is linear. If *T* is one-to-one, then there is at most one **x** such that  $T(\mathbf{x}) = \mathbf{0}$ .



# Proof

We have that  $T(\mathbf{0}) = \mathbf{0}$  since *T* is linear. If *T* is one-to-one, then there is at most one **x** such that  $T(\mathbf{x}) = \mathbf{0}$ . So the equation  $T(\mathbf{x}) = \mathbf{0}$  has only the trivial solution.



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### Example



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### Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if T is one-to-one.





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*T* is one-to-one if and only if the equation  $A\mathbf{x} = \mathbf{0}$  only has the trivial solution.



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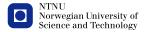
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Since *A* does not have a pivot position in every column, the equation  $A\mathbf{x} = \mathbf{0}$  has a free position and therefore a nontrivial solution, so *T* is not one-to-one.





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The following procedure outlines how to determine if a linear transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is one-to-one:



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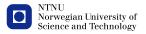
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- If A has a pivot position in every column, then the equation Ax = 0 has no free variable, so the equation Ax = 0 only has the trivial solution, and T is one-to-one.



### **Onto transformations**



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## **Onto transformations**

#### Definition of onto transformations

A transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is said to be *onto* (or *surjective*) if each **b** in  $\mathbb{R}^m$  is the image of at least one **x** in  $\mathbb{R}^n$ .



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Notice that a transformation T is onto if and only if the image of T is all of  $\mathbb{R}^m$ .



### Example



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### Example

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*T* is onto if and only if the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for all **b** in  $\mathbb{R}^3$ .



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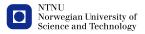
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We see that A has a pivot position in every row, so T is onto.





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# Onto and one-to-one linear transformations



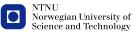
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# Onto and one-to-one linear transformations

#### Theorem 12

Let  $T : \mathbb{R}^n \to \mathbb{R}^m$  be a linear transformation and let A be the standard matrix for T.

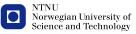


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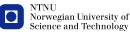


# Onto and one-to-one linear transformations

#### Theorem 12

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# Onto and one-to-one linear transformations

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- T maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$  if and only if the columns of A span  $\mathbb{R}^m$ .
- T is one-to-one if and only if the columns of A are linearly independent.



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T is onto



## *T* is onto $\iff$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $b \in \mathbb{R}^m$



# $\begin{array}{ll} T \text{ is onto} & \Longleftrightarrow \text{ the equation } A\mathbf{x} = \mathbf{b} \text{ has a solution} \\ & \text{ for every } b \in \mathbb{R}^m \\ & \Longleftrightarrow \text{ the columns of } A \text{ span } \mathbb{R}^m. \end{array}$



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#### T is one-to-one



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## *T* is one-to-one $\iff$ the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution



# $\begin{array}{ll} T \text{ is onto} & \Longleftrightarrow \text{ the equation } A\mathbf{x} = \mathbf{b} \text{ has a solution} \\ & \text{ for every } b \in \mathbb{R}^m \\ & \Longleftrightarrow \text{ the columns of } A \text{ span } \mathbb{R}^m. \end{array}$

## T is one-to-one $\iff$ the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

 $\iff$  the columns of A are

linearly independent.



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### Example



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### Example

Let  $T : \mathbb{R}^2 \to \mathbb{R}^3$  be a linear transformation such that  $T(x_1, x_2) = (x_1 + 2x_2, 2x_1 + x_2, 0)$ . Let us determine if T is onto and if it is one-to-one.

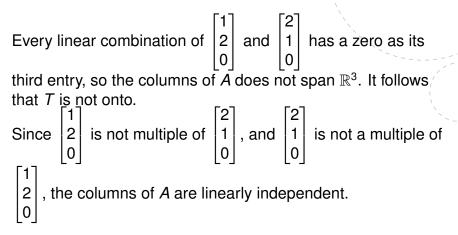


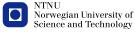
# Every linear combination of $\begin{bmatrix} 1\\ 2\\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2\\ 1\\ 0 \end{bmatrix}$ has a zero as its third entry, so the columns of *A* does not span $\mathbb{R}^3$ .

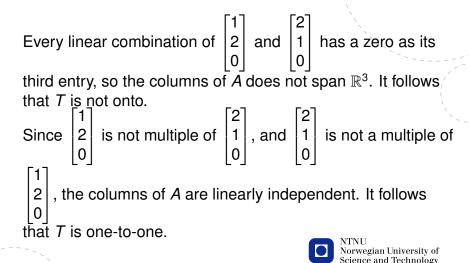


# Every linear combination of $\begin{bmatrix} 1\\ 2\\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2\\ 1\\ 0 \end{bmatrix}$ has a zero as its third entry, so the columns of *A* does not span $\mathbb{R}^3$ . It follows that *T* is not onto.









## **Tomorrow's lecture**

Tomorrow we shall

- look at applications of linear models,
- look at the use of Maple and WolframAlpha.

Section 1.10 in "Linear Algebras and Its Applications" (pages 80-90).

