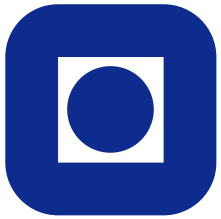

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TMA4115 - Calculus 3

Lecture 10, Feb 14

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday's lecture

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Yesterday's lecture

Yesterday we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,

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Yesterday's lecture

Yesterday we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector form*,

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Yesterday's lecture

Yesterday we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector form*,
- looked at applications of linear systems.

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Today's lecture

Today's lecture

We shall introduce and study *linear dependence* and *linear independence* of vectors.

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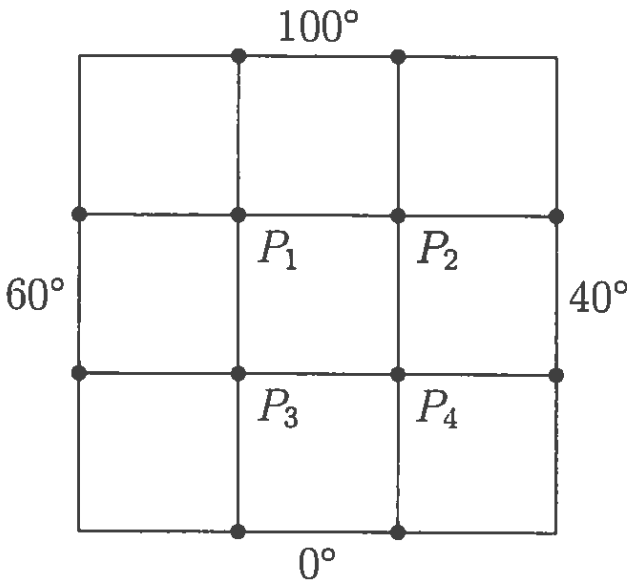
Problem 4 from December 2007

Problem 4 We shall determine the temperatures T_1, T_2, T_3, T_4 at four points P_1, P_2, P_3, P_4 on the square plate, see the figure on the right. Temperatures along the sides of the plate are shown on the figure.

Set up and solve a system of equations for T_1, T_2, T_3, T_4 if we assume that the temperature at each point P_1 to P_4 is the average of the temperatures at the four neighboring points (to the right, above, to the left, and below).



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Solution

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Solution

We have that

$$T_1 = \frac{1}{4}(60 + 100 + T_2 + T_3)$$
$$T_2 = \frac{1}{4}(T_1 + 100 + 40 + T_4)$$
$$T_3 = \frac{1}{4}(60 + T_1 + T_4 + 0)$$
$$T_4 = \frac{1}{4}(T_3 + T_2 + 40 + 0)$$

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Solution (cont.)

which is equivalent to

$$
\begin{aligned}
4T_1 - T_2 - T_3 &= 160 \\
-T_1 + 4T_2 - T_4 &= 140 \\
-T_1 + 4T_3 - T_4 &= 60 \\
-T_2 - T_3 + 4T_4 &= 40
\end{aligned}
$$

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Solution (cont.)

which is equivalent to

$$\begin{aligned} 4T_1 - T_2 - T_3 &= 160 \\ -T_1 + 4T_2 - T_4 &= 140 \\ -T_1 + 4T_3 - T_4 &= 60 \\ -T_2 - T_3 + 4T_4 &= 40 \end{aligned}$$

We reduce the augmented matrix of the system to its reduced echelon form.

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Solution (cont.)

$$\left[\begin{array}{rrrrr} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{array}\right]$$

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Solution (cont.)

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ -1 & 4 & 0 & -1 & 140 \\ 4 & -1 & -1 & 0 & 160 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix}$$

Solution (cont.)

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ -1 & 4 & 0 & -1 & 140 \\ 4 & -1 & -1 & 0 & 160 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 4 & -4 & 0 & 80 \\ 0 & -1 & 15 & -4 & 400 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix}$$

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Solution (cont.)

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ -1 & 4 & 0 & -1 & 140 \\ 4 & -1 & -1 & 0 & 160 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 4 & -4 & 0 & 80 \\ 0 & -1 & 15 & -4 & 400 \\ 0 & -1 & -1 & 4 & 40 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & -1 & 15 & -4 & 400 \\ 0 & 4 & -4 & 0 & 80 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 16 & -8 & 360 \\ 0 & 0 & -8 & 16 & 240 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 16 & -8 & 360 \\ 0 & 0 & -8 & 16 & 240 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 16 & -8 & 360 \end{bmatrix}$$

Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 16 & -8 & 360 \\ 0 & 0 & -8 & 16 & 240 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 16 & -8 & 360 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 0 & 24 & 840 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 16 & -8 & 360 \\ 0 & 0 & -8 & 16 & 240 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 16 & -8 & 360 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 0 & 24 & 840 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 1 & -60 \\ 0 & 1 & 1 & -4 & -40 \\ 0 & 0 & 1 & -2 & -30 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 & -95 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 & -95 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 65 \\ 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 & -95 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 65 \\ 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 35 \end{bmatrix}$$

We see that $T_1 = 65°$, $T_2 = 60°$, $T_3 = 40°$ and $T_4 = 35°$.

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Linear independence

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Linear independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

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Linear dependence

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Linear dependence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly dependent* if there exist weights $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Let us determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Let us determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

We must determine if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$$

has a nontrivial solution or not.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Let us determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

We must determine if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$$

has a nontrivial solution or not.

We write down the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}]$ of the equation and reduce it to echelon form.

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Solution

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}] =$$

Solution

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}] = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$

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Solution

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}] = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix}$$

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Solution

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}] = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}] = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the system has a free variable, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

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Linear independence of matrix columns

Linear independence of matrix columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_n]$ instead of a set of vectors.

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Linear independence of matrix columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

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Linear independence of matrix columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Thus we have the following important fact.

Linear independence of matrix columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Thus we have the following important fact.

Linear independence of matrix columns

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

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Example

Let us determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$ are linearly independent.

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Example

Let us determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$ are linearly independent.

We must determine if the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution or not.

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Example

Let us determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$ are linearly independent.

We must determine if the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution or not. We write down the augmented matrix $[A\ \mathbf{0}]$ of the equation and reduce it to echelon form.

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Solution

$[A\ \mathbf{0}]$

Solution

$$[A\,\mathbf{0}] = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

Solution

$$[A\,\mathbf{0}] = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

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Solution

$$[A\,\mathbf{0}] = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix}$$

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Solution

$$[A\,\mathbf{0}] = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

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Solution

$$[A\,\mathbf{0}] = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

We see that there are no free variables so $\mathbf{x} = 0$ is the only solution and the columns of A are therefore linearly independent.

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Linear dependence of sets containing the zero vector

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Linear dependence of sets containing the zero vector

Theorem 9

If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

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Linear dependence of sets containing the zero vector

Theorem 9

If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof:

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Linear dependence of sets containing the zero vector

Theorem 9

If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof: If $\mathbf{v}_j = \mathbf{0}$, then $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ when $c_j = 1$ and $c_i = 0$ for $i \neq j$.

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Linear dependence of sets containing the zero vector

Theorem 9

If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof: If $\mathbf{v}_j = \mathbf{0}$, then $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ when $c_j = 1$ and $c_i = 0$ for $i \neq j$. This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

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Sets of one vector

A set containing only one vector $\mathbf{v}$, is linearly independent if and only if $\mathbf{v}$ is not the zero vector.

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Sets of two vectors

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Sets of two vectors

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.

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Sets of two or more vectors

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Sets of two or more vectors

Theorem 7: Characterization of linearly dependent sets

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

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Sets of two or more vectors

Theorem 7: Characterization of linearly dependent sets

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
In fact, if S is linearly dependent and $\mathbf{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Proof of Theorem 7

Suppose there exist an index j and scalars $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_p$ such that

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \ldots c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \ldots c_p\mathbf{v}_p.$$

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Proof of Theorem 7

Suppose there exist an index j and scalars $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_p$ such that

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \ldots c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \ldots c_p\mathbf{v}_p.$$

If we let $c_j = -1$, then

$$c_1\mathbf{v}_1 + \ldots c_j\mathbf{v}_j = \mathbf{0}$$

and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$.
Suppose $\mathbf{v}_1 \neq \mathbf{0}$.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$.
Suppose $\mathbf{v}_1 \neq \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_p$ which are not all equal to zero such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$.
Suppose $\mathbf{v}_1 \neq \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_p$ which are not all equal to zero such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Let j be the largest subscript such that $c_j \neq 0$.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$.
Suppose $\mathbf{v}_1 \neq \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_p$ which are not all equal to zero such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Let j be the largest subscript such that $c_j \neq 0$. Notice that $j > 1$ because otherwise $c_1\mathbf{v}_1 = \mathbf{0}$ which cannot be the case since $\mathbf{v}_1 \neq \mathbf{0}$ and $c_1 \neq 0$.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$.
Suppose $\mathbf{v}_1 \neq \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_p$ which are not all equal to zero such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Let j be the largest subscript such that $c_j \neq 0$. Notice that $j > 1$ because otherwise $c_1\mathbf{v}_1 = \mathbf{0}$ which cannot be the case since $\mathbf{v}_1 \neq \mathbf{0}$ and $c_1 \neq 0$.
We then have that $c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j = \mathbf{0}$ from which it follows that $\mathbf{v}_j = \frac{-c_1}{c_j}\mathbf{v}_1 + \frac{-c_2}{c_j}\mathbf{v}_2 \cdots + \frac{-c_{j-1}}{c_j}\mathbf{v}_{j-1}$.

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Proof of Theorem 7 (cont.)

Assume then that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent. If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_p$. Suppose $\mathbf{v}_1 \neq \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_p$ which are not all equal to zero such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Let j be the largest subscript such that $c_j \neq 0$. Notice that $j > 1$ because otherwise $c_1\mathbf{v}_1 = \mathbf{0}$ which cannot be the case since $\mathbf{v}_1 \neq \mathbf{0}$ and $c_1 \neq 0$. We then have that $c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j = \mathbf{0}$ from which it follows that $\mathbf{v}_j = \frac{-c_1}{c_j}\mathbf{v}_1 + \frac{-c_2}{c_j}\mathbf{v}_2 \cdots + \frac{-c_{j-1}}{c_j}\mathbf{v}_{j-1}$. Thus, $\mathbf{v}_j$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Sets of two or more vectors

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.

Sets of two or more vectors

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent because

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent because
  1. $\mathbf{u}$ is not the zero vector,

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent because
  1. $\mathbf{u}$ is not the zero vector,
  2. $\mathbf{v}$ is not a linear combination of $\mathbf{u}$ ($\mathbf{v}$ is not a multiple of $\mathbf{u}$),

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent because
  1. $\mathbf{u}$ is not the zero vector,
  2. $\mathbf{v}$ is not a linear combination of $\mathbf{u}$ ($\mathbf{v}$ is not a multiple of $\mathbf{u}$),
  3. $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$ because $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$.

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Sets of two or more vectors

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof:

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$ and assume that $p > n$.

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$ and assume that $p > n$.
Let A be the $n \times p$ matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]$.

Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$ and assume that $p > n$. Let A be the $n \times p$ matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]$. Since $p > n$, not every column of A can be a pivot column, so there must be free variables and the equation $A\mathbf{x} = \mathbf{0}$ has therefore nontrivial solutions.

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$ and assume that $p > n$. Let A be the $n \times p$ matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]$. Since $p > n$, not every column of A can be a pivot column, so there must be free variables and the equation $A\mathbf{x} = \mathbf{0}$ has therefore nontrivial solutions. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

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Example

The following sets are all linearly dependent:

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Example

The following sets are all linearly dependent:

1. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$,

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Example

The following sets are all linearly dependent:

1. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ consists of 3 vectors in $\mathbb{R}^2$ and $3 > 2$.

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Example

The following sets are all linearly dependent:

1. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ consists of 3 vectors in $\mathbb{R}^2$ and $3 > 2$.
2. $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

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Example

The following sets are all linearly dependent:

1. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ consists of 3 vectors in $\mathbb{R}^2$ and $3 > 2$.
2. $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, because $\mathbf{v}_2 = \mathbf{0}$.

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Example

3. $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$,

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Example

3. $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, because $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_3$.

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Problem 6 from August 2010

For which values of the parameter a are the vectors $\mathbf{v}_1 = (1, -3, a)$, $\mathbf{v}_2 = (0, 1, a)$ and $\mathbf{v}_3 = (a, 2, 0)$ linearly dependent?

Solution

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Solution

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, if and only if the equation

$$\begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

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Solution

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, if and only if the equation

$$\begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

We reduce the augmented matrix $\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix}$ to an

echelon form.

Solution (cont.)

$$\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

$$\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & a & -a^2 & 0 \end{bmatrix}$$

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Solution (cont.)

$$\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & a & -a^2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & 0 & -2a-4a^2 & 0 \end{bmatrix}$$

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Solution (cont.)

$$\begin{bmatrix} 1 & 0 & a & 0 \\ -3 & 1 & 2 & 0 \\ a & a & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & a & -a^2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 2+3a & 0 \\ 0 & 0 & -2a-4a^2 & 0 \end{bmatrix}$$

We see that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent if and only if $-2a-4a^2 = -2a(1+2a) = 0$, that is, if and only if $a = 0$ or $a = -1/2$.

Problem 5 from December 2007

Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent vectors in $\mathbb{R}^n$. Are vectors

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_3,\ \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3$$

linearly dependent or linearly independent? (Remember to explain the answers.)

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Solution

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Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\mathbf{0} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3$$

Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3)\end{aligned}$$

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Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3.\end{aligned}$$

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Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3.\end{aligned}$$

It follows from the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$.

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Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3.\end{aligned}$$

It follows from the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$. The only solution of $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$ is $c_1 = c_2 = c_3$,

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Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3.\end{aligned}$$

It follows from the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$. The only solution of $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$ is $c_1 = c_2 = c_3$, so if $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$, then $c_1 = c_2 = c_3$.

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Solution

Suppose that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Then

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3.\end{aligned}$$

It follows from the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$. The only solution of $c_1 + c_2 = c_1 + c_3 = c_2 + c_3 = 0$ is $c_1 = c_2 = c_3$, so if $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$, then $c_1 = c_2 = c_3$. This shows that $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent.

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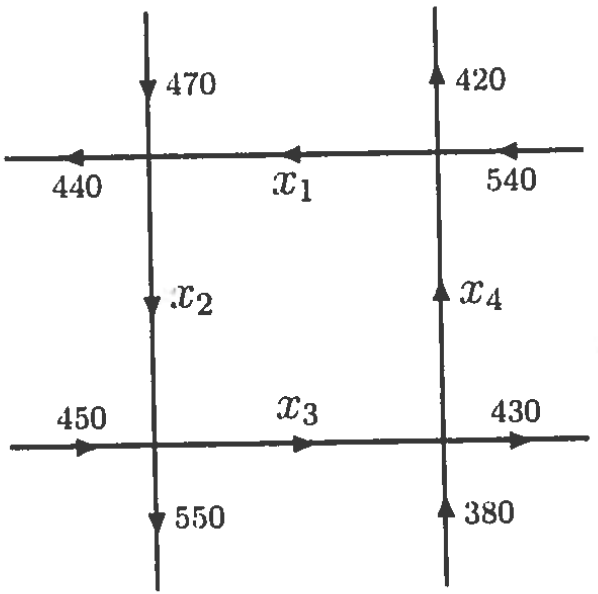
Problem 4 from December 2006

Problem 4 Four one-way streets in a city intesect as shown in the figure to the right. The number of cars passing per hour is shown in the figure.

Write down a system of equations in the form $A\mathbf{x} = \mathbf{b}$ which $\mathbf{x} = (x_1, x_2, x_3, x_4)$ must satisfy, and solve the system.

Find x_1, x_2 and x_3 if the x_4-section is closed for traffic, making $x_4 = 0$.



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Solution

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Solution

By looking at the number of cars going in and out of each of the 4 intersections, we see that the following equations must be satisfied:

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Solution

By looking at the number of cars going in and out of each of the 4 intersections, we see that the following equations must be satisfied:

$$
\begin{aligned}
x_1 + 470 &= 440 + x_2 \\
x_2 + 450 &= 550 + x_3 \\
x_3 + 380 &= 430 + x_4 \\
x_4 + 540 &= 420 + x_1
\end{aligned}
$$

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Solution (cont.)

This system of equations is equivalent to the equation $A\mathbf{x} = \mathbf{b}$

where $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -30 \\ 100 \\ 50 \\ -120 \end{bmatrix}$.

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Solution (cont.)

This system of equations is equivalent to the equation $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -30 \\ 100 \\ 50 \\ -120 \end{bmatrix}$.

We solve the system by reducing the augmented matrix $[A\ \mathbf{b}]$ to its reduced echelon form.

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Solution (cont.)

This system of equations is equivalent to the equation $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -30 \\ 100 \\ 50 \\ -120 \end{bmatrix}$.

We solve the system by reducing the augmented matrix $[A\ \mathbf{b}]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ -1 & 0 & 0 & 1 & -120 \end{bmatrix}$$

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Solution (cont.)

This system of equations is equivalent to the equation $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -30 \\ 100 \\ 50 \\ -120 \end{bmatrix}$.

We solve the system by reducing the augmented matrix $[A\ \mathbf{b}]$ to its reduced echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ -1 & 0 & 0 & 1 & -120 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & -1 & 0 & 1 & -150 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & -1 & 1 & -50 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & -1 & 1 & -50 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & -1 & 1 & -50 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & -1 & 1 & -50 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & -30 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 & 120 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solution (cont.)

We see that the system is consistent and that the general solution of it is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 120 \\ 150 \\ 50 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where t is a free parameter.

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Solution (cont.)

We see that the system is consistent and that the general solution of it is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 120 \\ 150 \\ 50 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where t is a free parameter.
If $x_4 = 0$, then $x_1 = 120$, $x_2 = 150$ and $x_3 = 50$.

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Problem 6 from August 2005

Let A be an $m \times n$ matrix where $m > n$. Explain why there is a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

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Solution

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Solution

The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if A has a pivot position in every row.

Solution

The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if A has a pivot position in every row. Since $m > n$, A has more rows that columns, and it can therefore not have a pivot position in every row (since the pivot positions appear in different columns).

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Solution

The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if A has a pivot position in every row. Since $m > n$, A has more rows that columns, and it can therefore not have a pivot position in every row (since the pivot positions appear in different columns). Thus, there is a vector $\mathbf{b} \in \mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

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Plan for next week

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Plan for next week

Wednesday we shall introduce and study

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations,

Section 1.8-1.9 in “Linear Algebras and Its Applications” (pages 62-80).

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Plan for next week

Wednesday we shall introduce and study

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations,

Section 1.8-1.9 in “Linear Algebras and Its Applications” (pages 62-80).

Thursday we shall look at *applications of linear models*. Section 1.10 in “Linear Algebras and Its Applications” (pages 80-90).

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