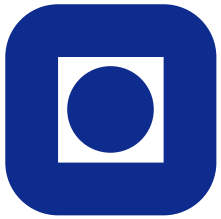

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TMA4115 - Calculus 3

Lecture 9, Feb 13

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Norwegian University of Science and Technology

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Review of last week's lecture

Last week we

- studied how to solve *systems of linear equations*,
- introduced *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations,
- introduced and studied *vectors*, *linear combinations* of vectors, subsets *spanned* by vectors, *vector equations*, the *product* of a matrix and a vector, *matrix equations*.

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Today's lecture

Today we shall

- introduce and solve *homogeneous* and *nonhomegeneous* matrix equations,
- learn how to write solution sets in *parametric vector form*,
- look at applications of linear systems.

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Matrix equations

Theorem 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n = \mathbf{b},$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n \ldots \mathbf{b}].$$

Note that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combinations of the columns of A, that is, if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

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Homogeneous linear systems

- A system of linear equations is said to be *homogeneous* if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely, $\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.
- This zero solution is usually called the *trivial solution*.
- The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

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Example

Let us describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

Parametric vector form

Whenever a solution set is written as

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors and $t_1, \ldots, t_p$ are parameters, we say that the solution is in *parametric vector form*.

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Example

Let us write the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form where

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

Solutions of nonhomogeneous systems

Let A be an $m \times n$ matrix and that $\mathbf{b}$ is a vector in $\mathbb{R}^m$. Suppose that $\mathbf{x} = \mathbf{p}$ is a solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$, and that the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is given by the parametric form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the solution set of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is given by the parametric form

$$\mathbf{x} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p.$$

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Example

Let us write the general solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

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Writing a solution set of a consistent system in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution **x** as a vector whose entries depend on the free variables, if any.
4. Decompose **x** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

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Example

Let us write the general solution to the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 - 3x_3 &= 2 \\ -x_1 + x_2 &= 0 \end{aligned}$$

in parametric vector form.

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Problem 3 from June 2005

Let $A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & -2 & -1 \\ 4 & -8 & 3 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ c \end{bmatrix}$ where c denotes an arbitrary real number.

1. Solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
2. For which values of c does the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ have a solution? Find the solution when the equation has a solution.

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Example

Suppose an economy consist of the Coal, Electric, and Steel sectors, and the output of each sector is distributed among the various sectors as shown in the following table.

Coal	Electric	Steel	Purchased by:
0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

If possible, let us find equilibrium prices that make each sector's income match its expenditures.

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Example

When propane burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O). Let us *balance* the chemical equation, that is find positive integers x_1 , x_2 , x_3, x_4 such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms are the same on the left match the corresponding numbers of atom on the right in the following equation.

$$x_1C_3H_8 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O$$

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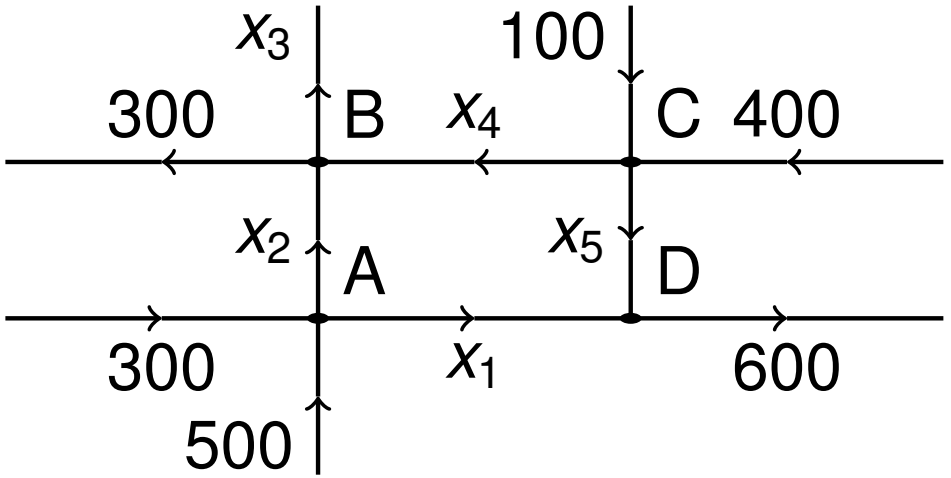
Example

The following network shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon.



Let us determine the general flow pattern for the network.

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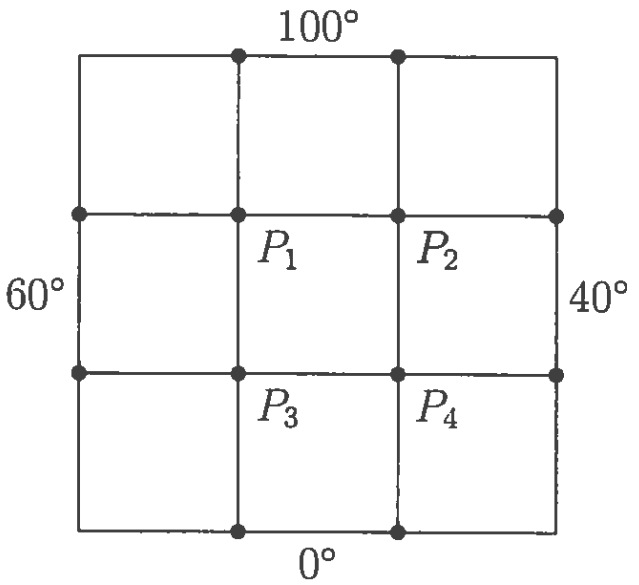
Problem 4 from December 2007

Problem 4 We shall determine the temperatures T_1, T_2, T_3, T_4 at four points P_1, P_2, P_3, P_4 on the square plate, see the figure on the right. Temperatures along the sides of the plate are shown on the figure.

Set up and solve a system of equations for T_1, T_2, T_3, T_4 if we assume that the temperature at each point P_1 to P_4 is the average of the temperatures at the four neighboring points (to the right, above, to the left, and below).



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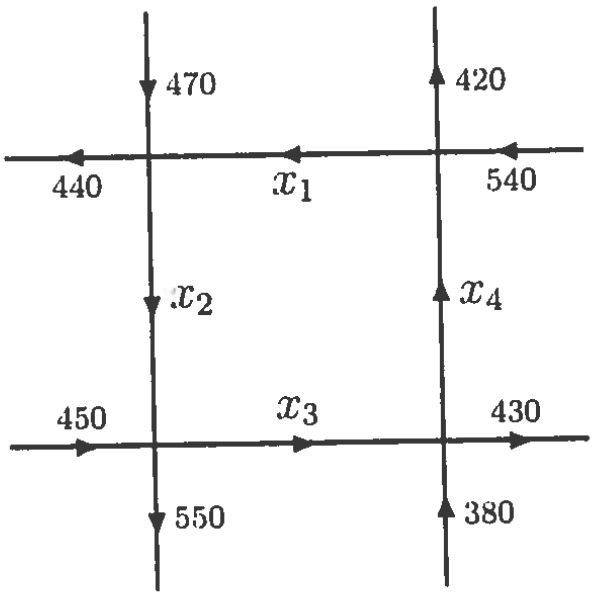
Problem 4 from December 2006

Problem 4 Four one-way streets in a city intesect as shown in the figure to the right. The number of cars passing per hour is shown in the figure.

Write down a system of equations in the form $A\mathbf{x} = \mathbf{b}$ which $\mathbf{x} = (x_1, x_2, x_3, x_4)$ must satisfy, and solve the system.

Find x_1, x_2 and x_3 if the x_4-section is closed for traffic, making $x_4 = 0$.



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Problem 6 from August 2005

Let A be an $m \times n$ matrix where $m > n$. Explain why there is a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

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Tomorrow's lecture

Tomorrow we shall introduce and study *linear dependence* and *linear independence* of vectors.
Section 1.7 in "Linear Algebras and Its Applications" (pages 55-62).

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