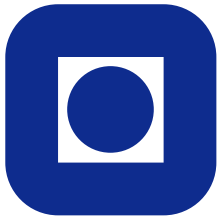

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TMA4115 - Calculus 3

Lecture 8, Feb 7

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Yesterday's lecture

Yesterday we

- studied how to solve *systems of linear equations*,
- introduced *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

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Today's lecture

Today we shall introduce and study

- *vectors*,
- *linear combinations* of vectors,
- subsets *spanned* by vectors,
- *vector equations*,
- the *product* of a matrix and a vector,
- *matrix equations*.

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Vectors

- A matrix with only one column is called a *column vector*, or simple a *vector*.
- An example of a vector with two entries is

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

  where v_1 and v_2 are real numbers.
- Let n be a positive integer. The set of all vectors with n real entries is denoted by $\mathbb{R}^n$.
- The set of all vectors with n complex entries is denoted by $\mathbb{C}^n$.

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Vectors

- Two vectors in $\mathbb{R}^n$ are equal if and only if their corresponding entries are equal.
- Given two vectors **u** and **v** in $\mathbb{R}^n$, their *sum* is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.
- Given a vector **u** and a real number c, the *scalar multiple* of **u** by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in **u** by c.

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Example

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Then

$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}.$$

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Geometric description of $\mathbb{R}^2$ and $\mathbb{R}^3$

- We can identify a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ from $\mathbb{R}^2$ with the point (a, b) in the real plane.
- We can identify a column vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from $\mathbb{R}^3$ with the point (a, b, c) in the real 3-dimensional space.

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Algebraic properties of $\mathbb{R}^n$

The vector whose entries are all zero is called the *zero vector* and is denoted by $\mathbf{0}$. For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

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Linear combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$ the vector $\mathbf{w}$ defined by

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ with weights $c_1, c_2, \ldots, c_p$. The weights in a linear combination can be any real numbers, including zero.

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Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Let us determine if $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}.$$

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Linear combinations

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2\ \ \ldots \mathbf{a}_p\ \mathbf{b}].$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ if and only if there exists a solution to the linear system corresponding to the matrix above.

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Subsets spanned by vectors

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and is called the subset *spanned* (or *generated*) by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
That is, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_p\mathbf{v}_p$$

where $c_1, c_2, \ldots, c_p$ are scalars.

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A geometric description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$. This set is the set of points on the line in $\mathbb{R}^2$ (or $\mathbb{R}^3$) through $\mathbf{v}$ and $\mathbf{0}$.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$ with $\mathbf{u}$ not a multiple of $\mathbf{v}$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$.

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Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$

Asking whether a vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix

$$[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p\ \mathbf{b}]$$

has a solution.
Note that Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

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Matrix-vector product

If A is an $m \times n$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{x}$ is in $\mathbb{R}^n$, then the *product of A and x*, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$, and that the number of entries in $A\mathbf{x}$, if it is defined, is equal to the numbers of rows in A.

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Example

The linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

is equivalent to the *vector equation*

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

and to the *matrix equation*

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

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Matrix equations

Theorem 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n = \mathbf{b},$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n \ldots \mathbf{b}].$$

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Matrix equations

Note that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combinations of the columns of A, that is, if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

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Example

Let $A \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let us determine for which values of b_1, b_2 and b_3 the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

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Matrix equations

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

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Example

Let us compute $A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

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Computations of $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vertex $\mathbf{x}$.

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Identities matrices

Let n be a positive integer. The matrix with 1s on the diagonal and 0s elsewhere is called an *identity matrix* and is denoted by I_n (or just I).

- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix I_3.
- $I_n\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$.

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Properties of the matrix-vector product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$.

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Example

Let $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$.

Let us determine if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

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Example

Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let us determine for which values of b_1, b_2 and b_3 the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$

Let us determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

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Problem 5 from August 2012

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

For which values of t does the equation $A\mathbf{x} = \mathbf{b}$ have none, one, or infinitely many solutions?

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Plan for next week

Wednesday we shall

- introduce and solve *homogeneous* and *nonhomegeneous* matrix equations,
- learn how to write solution sets in *parametric vector form*,
- look at applications of linear systems.

Section 1.5-1.6 in “Linear Algebras and Its Applications” (pages 43-55).

Thursday we shall introduce and study

- define and study *linear independence* of vectors.

Section 1.7 in “Linear Algebras and Its Applications” (pages 55-62).

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