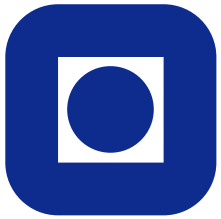

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TMA4115 - Calculus 3

Lecture 7, Feb 6

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Review of the previous lecture

Last time we

- looked at the method of *variation of parameters*,
- studied *forced harmonic motions*.

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Today's lecture

Today we shall

- study how to solve *systems of linear equations*,
- introduce *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

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Linear equations

A *linear equation* in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.

Examples

- The equations $4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$ are both linear.
- The equations $4x_1 - 5x_2 = x_1x_2$ and $x_2 = 2\sqrt{x_1} - 6$ are not linear.

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Systems of linear equations

- A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations involving the same variables.
- A *solution* of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.
- The set of all possible solutions is called the *solution set* of the linear system.
- Two linear systems are called *equivalent* if they have the same solution set.

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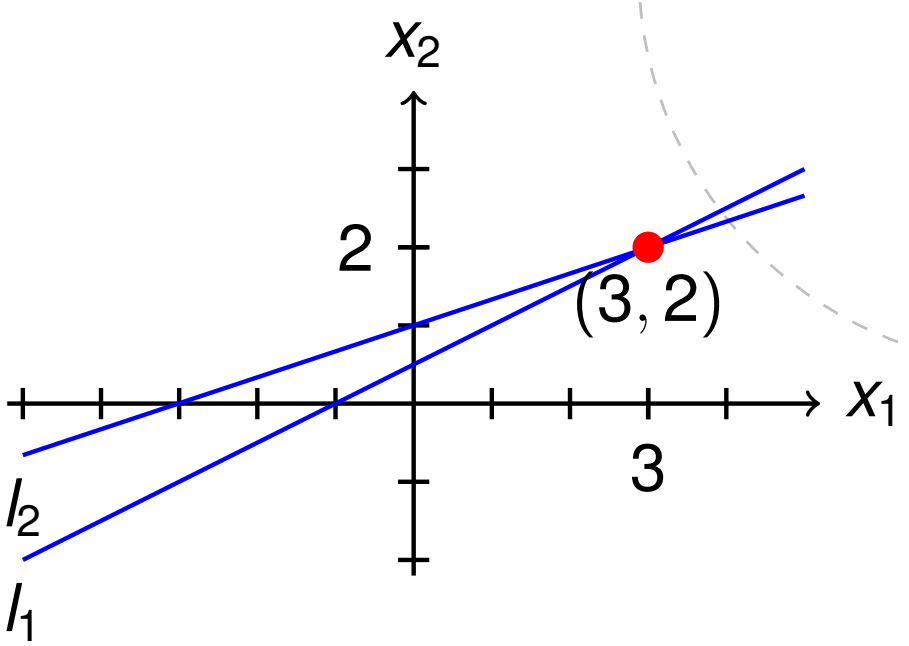
Example

$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$

is a system of linear equations.
$(3, 2)$ is a solution of the system because

$$3 - 2 \cdot 2 = -1$$
$$-3 + 3 \cdot 2 = 3$$

$(3, 2)$ is the only solution of the system, so $\{(3, 2)\}$ is the solution set of the system.



The graphs of the 2 equations are lines which we denote by l_1 and l_2. A pair of numbers (x_1, x_2) satisfies both equations if and only if the point (x_1, x_2) lies on both l_1 and l_2.

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Solutions to systems of linear equations

- A system of linear equations has
  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions.
- A system of linear equations is said to be *consistent* if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be *inconsistent* if it has no solution.

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Solving a linear system

The basic strategy for solving a linear system is to replace the system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

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Elementary row operations

Elementary row operations include the following:

1. (Replacement) Replace one equation by the sum of itself and a multiple of another equation.
2. (Interchange) Interchange two equations.
3. (Scaling) Multiply all coefficients in an equation by a nonzero constant.

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Elementary row operations

- Two linear systems are called *row equivalent* if there is a sequence of elementary row operations that transforms one system into the other.
- It is important to note that row operations are reversible.
- If two linear systems are row equivalent, then they have the same solution set.
- Our strategy for solving a linear system is to use elementary row operations to transform the system into a system which is elementary to solve.

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Matrix notation

- The essential information of a linear system can be recorded compactly in a rectangular array called a *matrix*.
- For the following system of equations,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the *coefficient matrix* of the system.

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Matrix notation

- An *augmented matrix* of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.
- For the following system of equations,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the *augmented matrix* of the system.

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Matrix size

- The size of a matrix tells how many rows and columns it has. If m and n are positive numbers, an $m \times n$ *matrix* is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)
- If a system of linear equations consists of m equations and has n variables, then the coefficient matrix of the system is an $m \times n$ matrix, and the augmented matrix of the system is an $m \times (n + 1)$ matrix.

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Elementary row operations

Elementary row operations include the following:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

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Elementary row operations

- Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other.
- It is important to note that row operations are reversible.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Our strategy for solving a linear system is to use elementary row operations to transform the augmented matrix of the system into a more elementary matrix.

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Echelon form

- A *nonzero* row (or column) in a matrix is a row (or column) that contains at least one nonzero entry.
- A *leading entry* of a row is the leftmost nonzero entry in a nonzero row.
- A rectangular matrix is in *echelon form* (or *row echelon form*) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

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Reduced echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or *reduced row echelon form*):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Echelon form

- An *echelon matrix* (respectively, *reduced echelon matrix*) is one that is in echelon form (respectively, reduced echelon form).

The following matrices are in echelon form. The leading entries ■ may have any nonzero value; the starred entries $*$ may have any value (including 0).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

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Reduced echelon form

The following matrices are in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

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Row reduction

Any nonzero matrix may be *row reduced* (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

Thm. 1: Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix U, we call U *an echelon form* (or *row echelon form*) of A; if U is in reduced echelon form, we call U *the reduced echelon form* of A.

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Pivot position

- Since the reduced echelon form of a matrix A is unique, the leading entries are always in the same positions in any echelon form of A.
- A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position.

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Row reduction algorithm

Let us apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

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Row reduction algorithm

Step 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

We interchange rows 1 and 3.

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Row reduction algorithm

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3

Use row replacement operations to create zeros in all positions below the pivot.

We add -1 times row 1 to row 2.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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Row reduction algorithm

Step 4

Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1—3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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Row reduction algorithm

For step 3, we add $-3/2$ times row 2 to row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row. Steps 1—3 require no work for this submatrix, and we have reached an echelon form of the full matrix.

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Row reduction algorithm

We perform one more step to obtain the reduced echelon form.

Step 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

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Row reduction algorithm

The rightmost pivot is in row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

We create zeros above it by adding -2 times row 3 to row 2, and -6 times row 3 to row 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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Row reduction algorithm

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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Row reduction algorithm

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is the reduced echelon form of the original matrix.

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Row reduction algorithm

The combination of steps 1—4 is called the *forward phase* of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the *backward phase*. The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

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Solutions of linear systems

Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 variables because the augmented matrix has four columns.

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Solutions of linear systems

The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called *basic variables*. The other variable, x_3, is called a *free variable*.

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Solutions of linear systems

- Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.
- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

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Solutions of linear systems

In the example

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

we solve the first and second equations for x_1 and x_2 (we ignore the third equation; it offers no restriction on the variables).

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Solutions of linear systems

We then get

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
$$x_3 \text{ is free}$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas above determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is (1,4,0); when $x_3 = 1$, the solution is (6,3,1).

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Parametric description of solution sets

- The description

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
$$x_3 \text{ is free}$$

  is a parametric description of solutions sets in which the free variables act as parameters.
- Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

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Parametric description of solution sets

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. As an example, let us consider the linear system

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

whose augmented matrix is the matrix

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we considered before.

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Parametric description of solution sets

If we add 5 times equation 2 to equation 1, we obtain the following equivalent system.

$$\begin{aligned} x_1 + 5x_2 &= 21 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.

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Inconsistent systems

When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

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Existence and uniqueness theorem

Theorem 2: Existence and uniqueness theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 \dots 0 & b \end{bmatrix}$$

with b nonzero.
If a linear system is consistent, then the solution set contains either

1. a unique solution, when there are no free variables, or
2. infinitely many solutions, when there is at least on free variable.

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Solving a linear system

To use row reduction to solve a linear system, we do the following:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

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Plan for tomorrow

Tomorrow we shall introduce and study

- *vectors*,
- *linear combinations of vectors*,
- subsets *spanned* by vectors,
- *vector equations*,
- the *product* of a matrix and a vector,
- *matrix equations*.

Section 1.3-1.4 in “Linear Algebras and Its Applications” (pages 24-42).

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