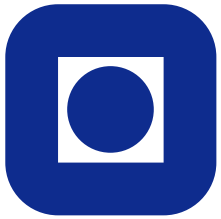

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TMA4115 - Calculus 3

Lecture 6, Jan 31

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Norwegian University of Science and Technology

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Review of yesterday's lecture

Yesterday we

- studied harmonic motions,
- studied solutions of second-order linear inhomogeneous differential equations,
- looked at the method of undetermined coefficients.

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Today's lecture

Today we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

General solutions to inhomogeneous equations

If y_p is a particular solution to the inhomogeneous equation $y'' + py' + qy = f$ and y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$, then the general solution to the inhomogeneous equation $y'' + py' + qy = f$ is

$$y = y_p + c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are constants.

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The method of undetermined coefficients

Consider the inhomogeneous second-order linear differential equation

$$y'' + py' + qy = f.$$

If the function f has a form that is replicated under differentiation, then look for a solution with the same general form as f.

Forced harmonic motion

We will now apply the technique of undetermined coefficients to analyze harmonic motion with an external sinusoidal forcing term.
The equation we need to solve is

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

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Forced undamped harmonic motion

Let us first assume that $c = 0$. Then the equation becomes

$$y'' + \omega_0^2 y = A\cos(\omega t).$$

The general solution to the homogeneous solution $y'' + \omega_0^2 y = 0$ is $y_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$.
Let us find a particular solution by using the method of undetermined coefficients. We will first look at the case where $\omega \neq \omega_0$, and then at the case where $\omega = \omega_0$.

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The case $\omega \neq \omega_0$

If $\omega \neq \omega_0$, then the general solution to

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$$

is $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t)$.
Let us look at the solution where the motion starts at equilibrium. This means that $y(0) = y'(0) = 0$. We then have that $0 = y(0) = c_1 + \frac{A}{\omega_0^2 - \omega^2}$ and $0 = y'(0) = c_2\omega_0$, so

$$y(t) = \frac{A}{\omega_0^2 - \omega^2}\big(\cos(\omega t) - \cos(\omega_0 t)\big).$$

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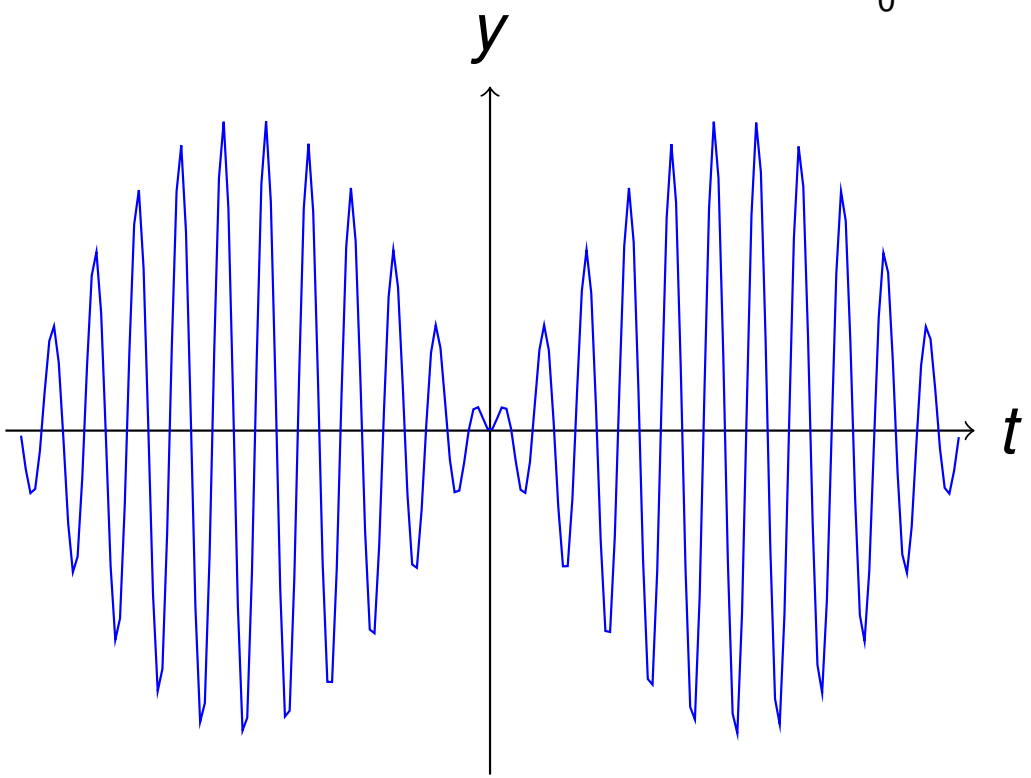
The case $\omega \neq \omega_0$

$$y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$$



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The case $\omega \neq \omega_0$

Consider the solution $y(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$.

Let $\overline{\omega} = (\omega_0 + \omega)/2$ and $\delta = (\omega_0 - \omega)/2$. $\overline{\omega}$ is called the *mean frequency*, and δ is called the *half difference*. We then have that

$$\begin{aligned} y(t) &= \frac{A}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t)) \\ &= \frac{A}{4\overline{\omega}\delta}(\cos((\overline{\omega} - \delta)t) - \cos((\overline{\omega} + \delta)t)) \\ &= \frac{A\sin(\delta t)}{2\overline{\omega}\delta}\sin(\overline{\omega} t). \end{aligned}$$

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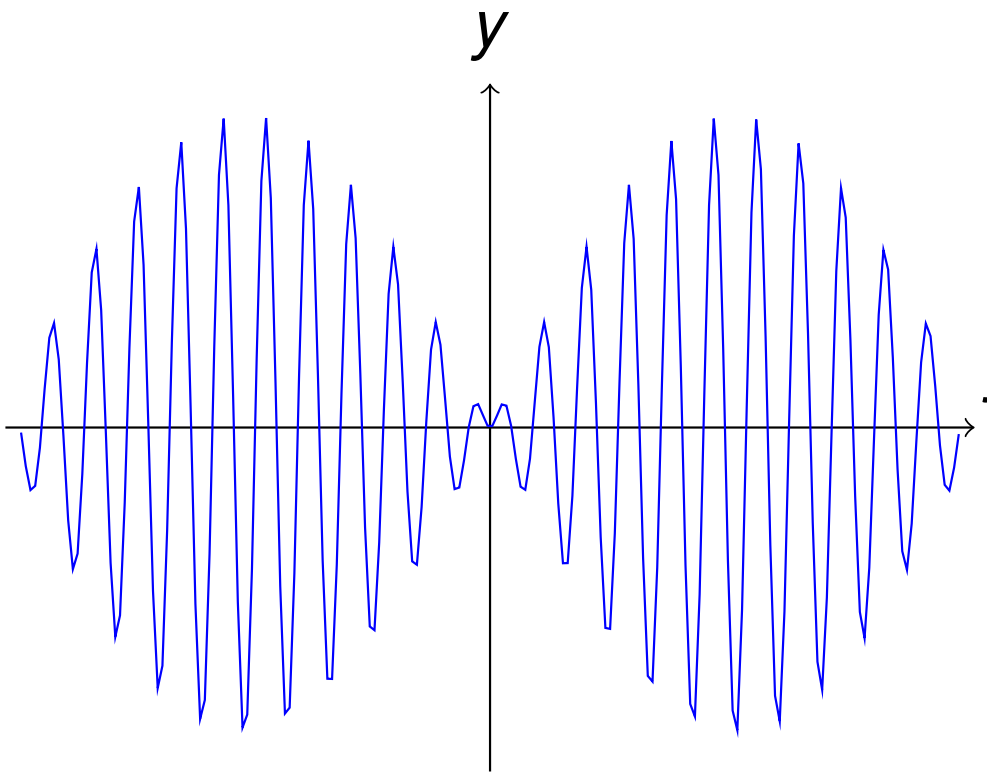
The case $\omega \neq \omega_0$



$$y(t) = \frac{A\sin(\delta t)}{2\overline{\omega}\delta}\sin(\overline{\omega}t)$$

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The case $\omega = \omega_0$

If $\omega \neq \omega_0$, then the general solution to

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$$

is $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{A}{2\omega_0} t \sin(\omega_0 t)$.
Let us look at the solution where the motion starts at equilibrium. This means that $y(0) = y'(0) = 0$. We then have that $0 = y(0) = c_1$ and $0 = y'(0) = c_2\omega_0$, so

$$y(t) = \frac{A}{2\omega_0} t \sin(\omega_0 t).$$

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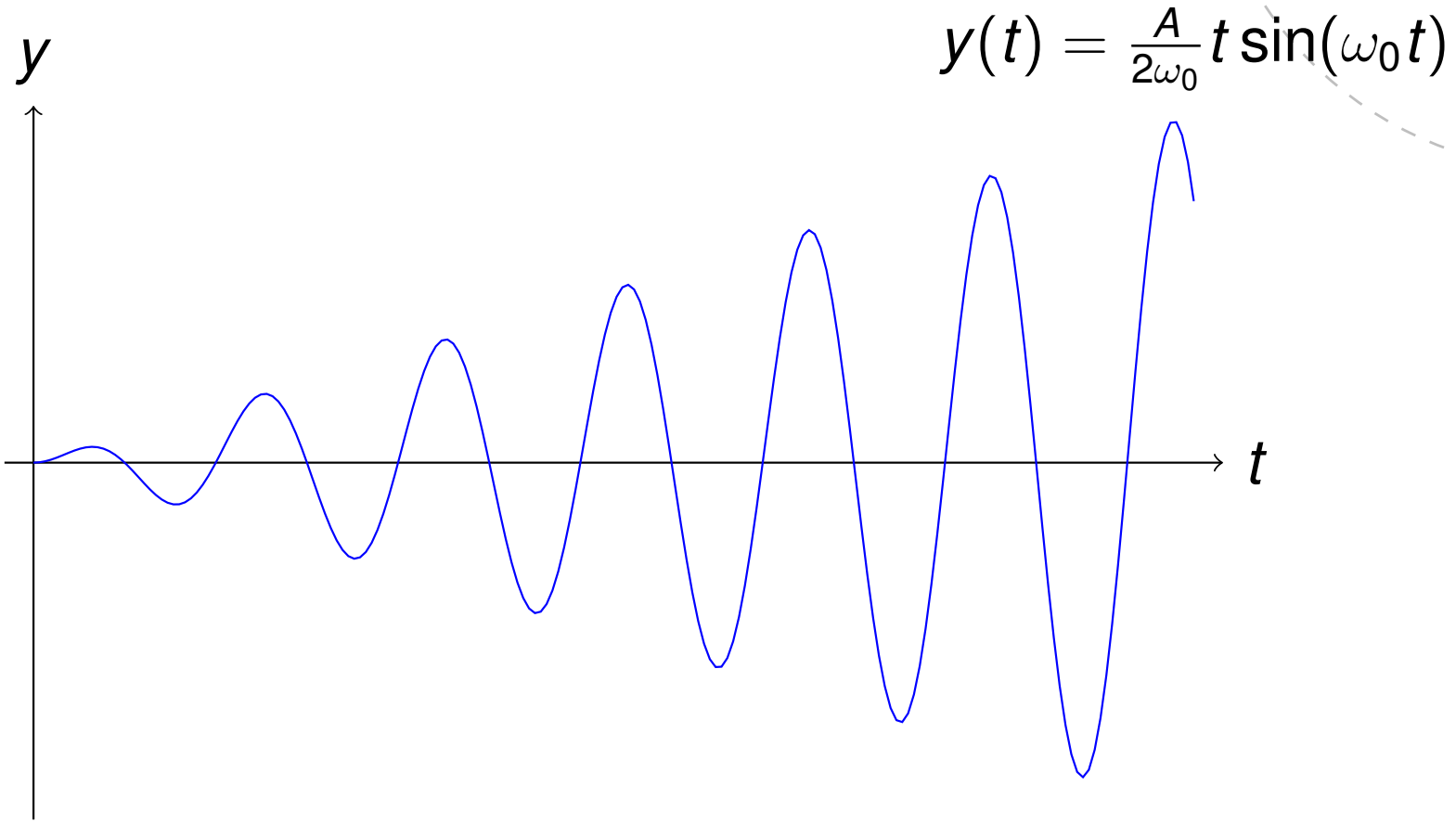
The case $\omega = \omega_0$



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The forced damped harmonic motion

If we add a damping term to the system we get the equation

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

Let us assume that $c < \omega_0$. Then the general solution to the homogeneous equation $y'' + 2cy' + \omega_0^2 y = 0$ is

$y_h(t) = e^{-ct}(c_1 \cos(\eta t) + c_2 \sin(\eta t))$ where $\eta = \sqrt{\omega_0^2 - c^2}$.

To find a particular solution we will use the technique of undetermined coefficients and the complex method. This means that we will be looking for a solution $z(t) = ae^{i\omega t}$ to the equation $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$.

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The forced damped harmonic motion

The general solution to

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

is

$$y(t) = e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t)) + \frac{A}{R}\cos(\omega t - \phi)$$

where $R = |P(i\omega)|$ and $\phi = \mathrm{Arg}(P(i\omega))$.

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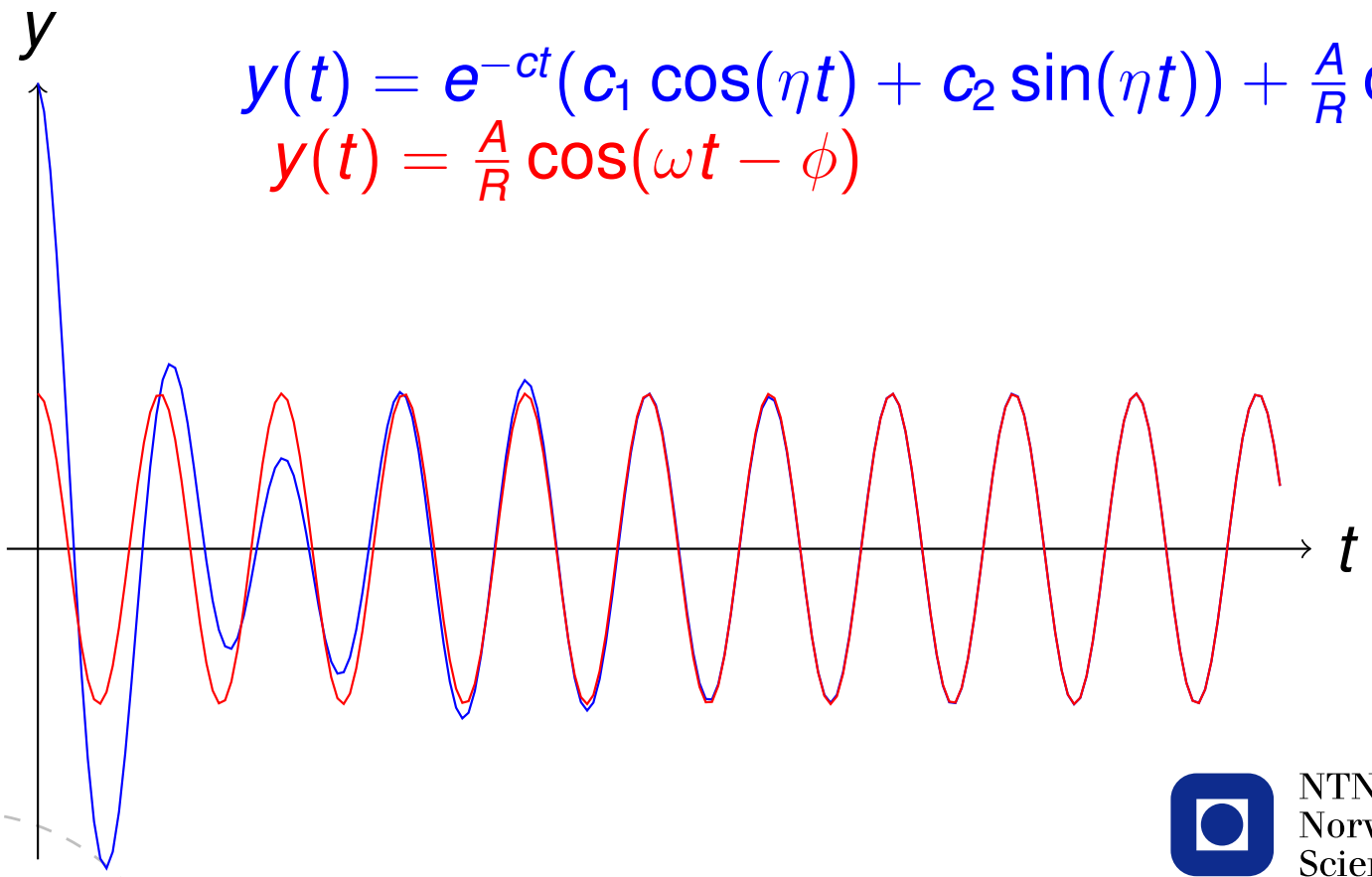
The forced damped harmonic motion



$$y(t) = e^{-ct}(c_1 \cos(\eta t) + c_2 \sin(\eta t)) + \frac{A}{R}\cos(\omega t - \phi)$$

$$y(t) = \frac{A}{R}\cos(\omega t - \phi)$$

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Steady-state and transient terms

The general solution to

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t).$$

is $y(t) = e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t)) + \frac{A}{R}\cos(\omega t - \phi)$.
The term $e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t))$ is called the *transition term*, and the term $\frac{A}{R}\cos(\omega t - \phi)$ is called the *steady-state* term.

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Variation of parameters

We are looking for a particular solution to an inhomogeneous second-order linear differential equation

$$y'' + p(t)y' + q(t)y = f(t).$$

Suppose that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$. The idea behind the *variation of parameters* method is to look for a particular solution of the form $y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ where v_1 and v_2 are unknown functions.

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Variation of parameters

To find a particular solution to $y'' + py' + qy = f$ using the method of variation of parameters we follow these steps.

1. Find a fundamental set of solutions y_1, y_2 to the homogeneous equation $y'' + py' + qy = 0$.
2. Let $y_p = v_1 y_1 + v_2 y_2$ where v_1 and v_2 are functions to be determined.
3. Find v_1' and v_2' such that
$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = f.$$
4. Let $v_1(t) = \int v_1'(t)\, dt$ and $v_2(t) = \int v_2'(t)\, dt$.
5. Substitute v_1 and v_2 into $y_p = v_1 y_1 + v_2 y_2$.

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Variation of parameters

If

$$v_1(t) = \int \frac{-y_2(t)f(t)}{W(t)}\,dt$$

and

$$v_2(t) = \int \frac{y_1(t)f(t)}{W(t)}\,dt$$

where $W(t)$ is the Wronskian of y_1 and y_2, then

$$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

is a solution of $y'' + py' + qy = f$.

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Plan for next week

Wednesday we shall

- study how to solve *systems of linear equations*,
- introduce *row reduction*, *echelon forms*, *pivot positions*, *the row reduction algorithm*, and *parametric descriptions* of solution sets of systems of linear equations.

Section 1.1-1.2 in “Linear Algebras and Its Applications” (pages 1-23).

Thursday we shall introduce and study

- *vectors*,
- *linear combinations* of vectors,
- subsets *spanned* by vectors,
- *vector equations*.

Section 1.3 in “Linear Algebras and Its Applications” (pages 24-34).

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