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TMA4115 - Calculus 3

Lecture 5, Jan 30

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Review of the previous lecture

Last time we

- studied second-order linear differential equations,
- introduced the *Wronskian*,
- completely solved second-order homogeneous linear differential equations with constant coefficients.

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Today's lecture

Today we shall

- study harmonic motions,
- study solutions of second-order linear inhomogeneous differential equations,
- look at the method of undetermined coefficients.

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Second-order homogeneous linear differential equations

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

on the interval (α, β). Then

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is the general solution of (1).

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Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,
  2. or by showing that the Wronskian of y_1 and y_2 is not zero at any point.

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Homogeneous equations with constant coefficients

Consider the second-order homogeneous linear differential equation

$$y'' + py' + qy = 0$$

with constant coefficients.

- The *characteristic polynomial* of the equation is the polynomial $\lambda^2 + p\lambda + q$.
- The roots

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

of $\lambda^2 + p\lambda + q$ are called the *characteristic roots* of the equation.

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Homogeneous equations with constant coefficients

- If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2, and the general solution of $y'' + py' + qy = 0$ is
$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$
- If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$, and the general solution of $y'' + py' + qy = 0$ is
$$y(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt).$$

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Homogeneous equations with constant coefficients

- If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root λ, and the general solution of $y'' + py' + qy = 0$ is

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

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Harmonic motion

The motion described by a solution to the equation

$$y'' + 2cy' + \omega_0^2 y = 0$$

where $c \geq 0$ and $\omega_0 > 0$, is called a *harmonic motion*.

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Simple harmonic motion

If $c = 0$ we say that the system is *undamped*. In that case, the equation becomes

$$y'' + \omega_0^2 y = 0$$

where $\omega_0 > 0$.
The general solution to this equation is

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

The motion described by this solution is called a *simple harmonic motion*. The number ω_0 is called the *natural frequency*. The number $T = 2\pi/\omega_0$ is called the *period*.

Amplitude and phase angle

It is frequently convenient to put the solution $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ into another form that is more convenient and more revealing of the nature of the solution. Let $A = |c_1 + c_2 i| = \sqrt{c_1^2 + c_2^2}$ and $\phi = \operatorname{Arg}(c_1 + c_2 i)$. Then

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = A \cos(\omega_0 t - \phi).$$

The number A is called the *amplitude*, and the number ϕ is called the *phase*.

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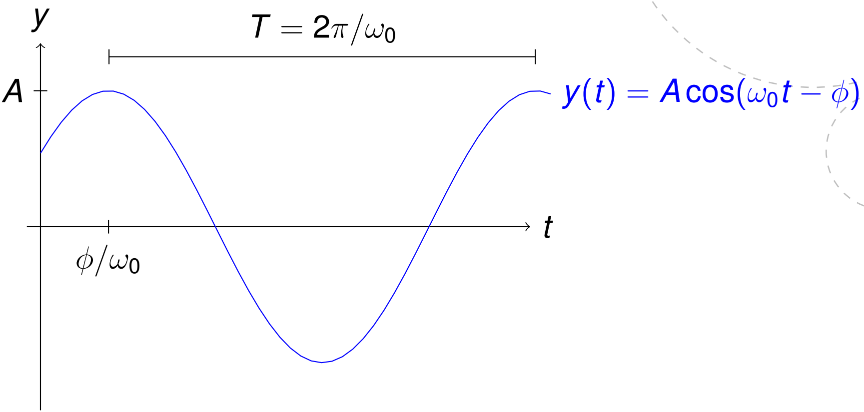
Simple harmonic motion



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The underdamped case

If $0 < c < \omega_0$, then the general solution to

$$y'' + 2cy' + \omega_0^2 y = 0$$

is

$$y(t) = e^{-ct}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$

where $\omega = \sqrt{\omega_0^2 - c^2}$.

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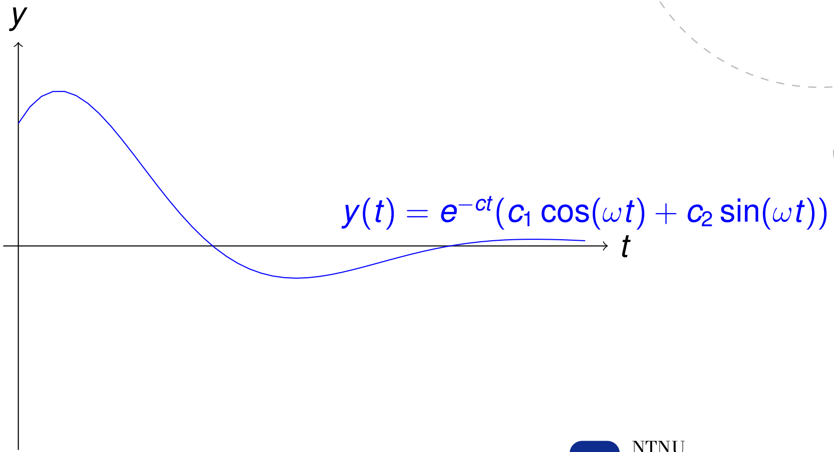
The underdamped case



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The overdamped case

If $c > \omega_0$, then the general solution to

$$y'' + 2cy' + \omega_0^2 y = 0$$

is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where $\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$ and $\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}$.

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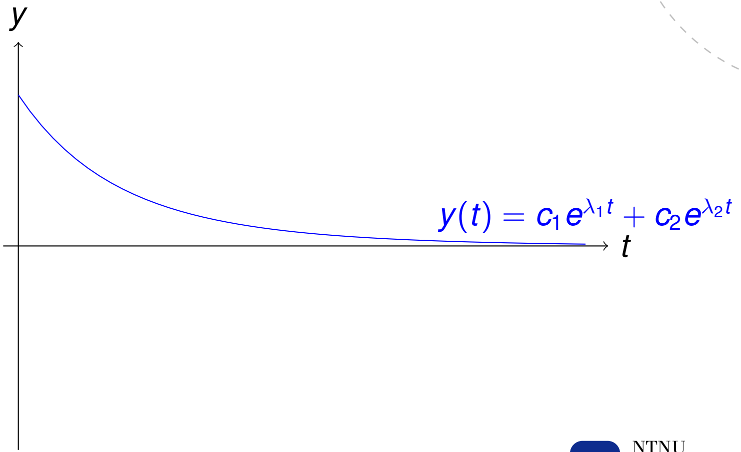
The overdamped case



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The critically damped case

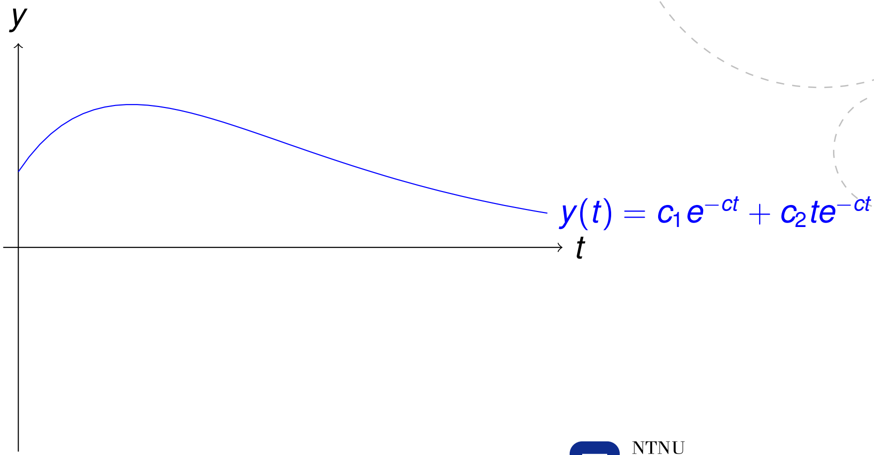
If $c = \omega_0$, then the general solution to

$$y'' + 2cy' + \omega_0^2 y = 0$$

is

$$y(t) = c_1 e^{-ct} + c_2 t e^{-ct}.$$

The critically damped case



Inhomogeneous equations

We now turn to the solution of inhomogeneous second-order linear differential equations

$$y'' + py' + qy = f \tag{2}$$

where $p = p(t)$, $q = q(t)$ and $f = f(t)$ are functions of the independent variable.

If y_p is a particular solution to (2), and y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$, then the general solution to (2) is

$$y = y_p + c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are constants.

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The method of undetermined coefficients

Consider the inhomogeneous second-order linear differential equation

$$y'' + py' + qy = f.$$

If the function f has a form that is replicated under differentiation, then look for a solution with the same general form as f.

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Exponential forcing terms

If $f(t) = e^{at}$, then $f'(t) = ae^{at}$, so we will look for a solution of the form $y(t) = be^{at}$.

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Trigonometric forcing terms

If

$$f(t) = A\cos(\omega t) + B\sin(\omega t),$$

then

$$f'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t),$$

so we will look for a solution of the form

$$y(t) = a\cos(\omega t) + b\sin(\omega t).$$

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The complex method

There is another way to find a particular solution in situations where

$$f(t) = A\cos(\omega t) + B\sin(\omega t).$$

If $z(t)$ is a solution of the equation

$$y'' + py' + qy = e^{\omega i t},$$

then a suitable linear combination of $\mathrm{Re}(z(t))$ and $\mathrm{Im}(z(t))$ will be a solution to

$$y'' + py' + qy = A\cos(\omega t) + B\sin(\omega t).$$

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Polynomial forcing terms

If

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots a_1 t + a_0,$$

then

$$f'(t) = n a_n t^{n-1} + (n-1) a_{n-1} t^{n-2} + \ldots a_1,$$

so we will look for a solution of the form

$$y(t) = b_n t^n + b_{n-1} t^{n-1} + \ldots b_1 t + b_0.$$

Exceptional cases

The method of undetermined coefficients looks straightforward. There are, however, some exceptional cases to look out for. If the forcing term f, and hence the proposed solution, is a solution to the homogeneous equation $y'' + py' + qy = 0$, then the proposed solution wouldn't work. Instead we have to multiply the proposed solution by t.

Combination forcing terms

If y_f is a solution the differential equation $y'' + py' + qy = f$, y_g is a solution the differential equation $y'' + py' + qy = g$, and c_1 and c_2 are constants, then

$$y(t) = c_1 y_f(t) + c_2 y_g(t)$$

is a solution to the differential equation

$$y'' + py' + qy = c_1 f + c_2 g.$$

Plan for tomorrow

Tomorrow we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

Section 4.6 and 4.7 in “Second-Order Equations” (pages pages lxxii–lxxxvi).

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