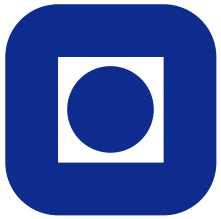

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TMA4115 - Calculus 3

Lecture 4, Jan 24

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Review of yesterday's lecture

Yesterday we

- looked at how to use complex numbers to solve polynomial equations,
- looked at *the fundamental theorem of algebra*,
- introduced *the complex exponential function*,
- and studied extensions of trigonometric functions to the complex numbers.

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Today's lecture

Today we shall

- study second-order linear differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.

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Second-order differential equations

A second-order differential equation is a differential equation which can be written on the form

$$y'' = f(t, y, y').$$

A solution to such an equation is a twice continuously differentiable function $y(t)$ satisfying

$$y''(t) = f(t, y(t), y'(t)).$$

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Examples of second-order differential equations

- $y'' + \cos(y) = e^t$.
- $y'' + 5y = 0$.
- $t^2 y'' + \sin(t) y = 3$.

Second-order linear differential equations

A second-order linear differential equation is a differential equation with can be written on the form

$$y'' + p(t)y' + q(t)y = g(t).$$

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$ is not linear.
- $y'' + 5y = 0$ is linear.
- $t^3 y'' + \sin(t) y = 3$ is linear.

Existence and uniqueness of solutions to a second-order linear equation

Suppose the functions p, q and g are continuous on the interval (α, β). Let t_0 be any point in (α, β). Then for any real numbers a and b there is one and only one function defined on (α, β) which is a solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

on (α, β) and satisfies the initial conditions $y(t_0) = a$ and $y'(t_0) = b$.

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Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$. If $g \neq 0$, then the differential equation is called *inhomogeneous* or *nonhomegeneous*.

Examples

- $y'' + 5y = 0$ is homogeneous.
- $t^3y'' + \sin(t)y = 3$ is inhomogeneous.

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The superposition principle

If y_1 and y_2 are solutions to the second-order homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then so is $y(t) = c_1 y_1(t) + c_2 y_2(t)$ for any choice of constants c_1 and c_2.

Linear combinations

- When y_1 and y_2 are functions and c_1 and c_2 are constants, then the function $c_1 y_1(t) + c_2 y_2(t)$ is a *linear combination* of y_1 and y_2.
- So the previous results says that if y_1 and y_2 are solutions to a second-order homogeneous linear differential equation, then any linear combination of y_1 and y_2 is also a solution to the same differential equation.

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The Wronskian

Let u and v be two differential functions. The *Wronskian* of u and v is the function

$$W(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t).$$

Solutions to initial value problems

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

on the interval (α, β). Let t_0 be a point in the interval (α, β) and let a and b be arbitrary real numbers. If $W(t)$ is the Wronskian of y_1 and y_2 and $W(t_0) \neq 0$, then there exist constants c_1 and c_2 such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the unique solution to (1) on (α, β) which satisfies $y(t_0) = a$ and $y'(t_0) = b$.

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Proposition 1.26

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then the Wronskian of y_1 and y_2 is either identically equal to zero on the interval (α, β), or it is never equal to zero on the interval (α, β).

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Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.
- u and v are *linear independent* if they are not linear dependent.
- If $u = cv$ for some constant c, then u and v are linear dependent because $u - cv = 0$.
- Conversely, if u and v are linear dependent, then $c_1 u + c_2 v = 0$ for some choice of constants c_1 and c_2 which are no both zero, and then $u = -(c_2/c_1)v$ if $c_1 \neq 0$, and $v = -(c_1/c_2)u$ if $c_2 \neq 0$.

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Linear dependent functions

Two functions u and v are linear dependent if and only if u is a constant multiple of v, or v is a constant multiple of u.

Proposition 1.27

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β).
Then y_1 and y_2 are linearly independent on (α, β) if and only if the Wronskian of y_1 and y_2 is identically equal to zero on the interval (a, b).

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Theorem 1.23

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

on the interval (α, β). Then any solution to the equation is a linear combination of y_1 and y_2.

Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,
  2. or by showing that the Wronskian of y_1 and y_2 is not zero at any point.

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Second-order homogeneous linear differential equations with constant coefficients

A second-order homogeneous linear differential equation with constant coefficients is a differential equation with can be written on the form

$$y'' + py' + qy = 0$$

where p and q are constants.

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Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

1. $p^2 - 4q > 0$,
2. $p^2 - 4q < 0$,
3. $p^2 - 4q = 0$.

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Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The general solution of

$$y'' + py' + qy = 0$$

is then

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2 which have the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ where a and b are real numbers and $b \neq 0$.

$$\begin{aligned} z(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ &= c_1 e^{at}(\cos(bt) + i\sin(bt)) + c_2 e^{at}(\cos(bt) + i\sin(bt)) \end{aligned}$$

is then a general solution of

$$y'' + py' + qy = 0,$$

and so is

$$y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt).$$

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Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The general solution of

$$y'' + py' + qy = 0$$

is then

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

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Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).
Thursday we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

Section 4.6 and 4.7 (pages lxxii–lxxxvi).

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