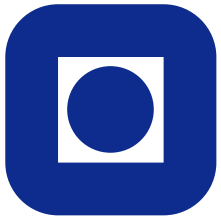

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TMA4115 - Calculus 3

Lecture 3, Jan 23

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Review of last week

Last week we

- introduced complex numbers, both in a geometric way and in an algebraic way,
- defined $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$ and $\arg(z)$ for a complex number z,
- defined addition and multiplication of complex numbers,
- defined complex conjugation,
- introduced polar representation of complex numbers,
- computed powers of complex numbers,
- defined and computed roots of complex numbers.

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Today's lecture

Today we shall

- use complex numbers to solve polynomial equations,
- look at *the fundamental theorem of algebra*,
- introduce *the complex exponential function*,
- and study extensions of trigonometric functions to the complex numbers.

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Solutions to second degree equations

If a, b, c are complex numbers and $a \neq 0$, then the solutions to the equation $az^2 + bz + c = 0$ are $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Problem 1 from the exam from June 2012

Solve $w^2 = (-1 + i\sqrt{3})/2$. Find all solutions of the equation $z^4 + z^2 + 1 = 0$ and draw them in the complex plane. Write the solutions in the form $x + iy$.

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Roots of polynomials

If $P(z)$ is a polynomial, then a solution to the equation $P(z) = 0$ is called a *root* (or *zero*) of $P(z)$.
z_0 is a root of $P(z)$ if and only if $(z - z_0)$ is a factor of $P(z)$ (i.e., if $P(z) = (z - z_0)Q(z)$ for some polynomial $Q(z)$).

The fundamental theorem of algebra

Every complex polynomial of degree 1 or higher has a least one complex root.

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Roots of real polynomials

If w is a root of a real polynomial $\sum_{k=0}^{n} a_k z^k$, then $\overline{w}$ is also a root of $\sum_{k=0}^{n} a_k z^k$.

Exercise 34 on page xxvii

Check that $z_1 = 1 - \sqrt{3}i$ is a zero of $P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16$, and find all the zeros of P.

Complex functions

A complex function f is a rule that assigns a unique complex number $f(z)$ to each number z in some set of complex numbers (called the *domain* of f).

Examples of complex functions

- $f(z) = \mathrm{Re}(z)$
- $g(z) = \mathrm{Im}(z)$
- $h(z) = |z|$
- $j(z) = \mathrm{Arg}(z)$
- $k(z) = \overline{z}$
- $p(z) = z^2 - 4z + 6$

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Graphic representations of complex functions

- We cannot draw the graph of a complex function since we would need 4 dimensions to do that.
- Instead, we can graphically represent the behavior of a complex function $w = f(z)$ by drawing the z-plane and the w-plane separately, and showing the image in the w-plane of certain, appropriately chosen set of points in the z-plane.

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Complex functions

Limits, continuity and differentiability of complex functions can be defined just as for real functions.

Examples of complex functions

- Every complex polynomial is differentiable, and hence continuous.
- The functions $f(z) = \mathrm{Re}(z)$, $g(z) = \mathrm{Im}(z)$, $h(z) = |z|$ and $k(z) = \overline{z}$ are continuous, but not differentiable.
- The function $j(z) = \mathrm{Arg}(z)$ is not continuous.

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The exponential function

One can show that the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely for all complex numbers z.

We denote the sum of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ as the exponential function e^z.

e^z is also the limit $\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$.

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The exponential function and cos and sin

If y is a real number, then

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iy)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{(2n+1)!} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} = \cos(y) + i \sin(y).$$

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The exponential function

One can show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$. It follows that $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$ for $z = x + iy$.

The exponential function and polar representation

If $z \neq 0$, then

$$\begin{aligned} z &= |z|(\cos(\arg(z)) + i\sin(\arg(z))) \\ &= e^{\ln(|z|)} e^{i\arg(z)} = e^{\ln(|z|)+i\arg(z)}. \end{aligned}$$

The exponential function is not injective (because $e^{x+iy} = e^{x+i(y+2\pi)}$), and does therefore not have an inverse.

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Properties of the exponential function

If $z = x + iy$, then

- $\overline{e^z} = e^{\bar{z}}$
- $\mathrm{Re}(e^z) = e^x \cos y$
- $\mathrm{Im}(e^z) = e^x \sin y$
- $|e^z| = e^x$
- $\arg(e^z) = y$

One can also show that e^z is differentiable and that $\frac{d}{dz} e^z = e^z$.

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Sine and cosine

One can show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ converge absolutely for all complex numbers z.

We denote the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ as $\cos(z)$, and the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ as $\sin(z)$.

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Properties of sin and cos

- If z is a complex number, then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

- sin and cos are periodic with period 2π.
- sin and cos are differentiable and $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$.

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Hyperbolic sine and cosine

One can show that the series $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ converge absolutely for all complex numbers z.

We denote the sum of $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ as $\cosh(z)$, and the sum of $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ as $\sinh(z)$.

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Properties of sinh and cosh

- If z is a complex number, then

$$\cosh z = \frac{e^z + e^{-z}}{2} \text{ and } \sinh z = \frac{e^z - e^{-z}}{2}.$$

- sinh and cosh are periodic with period $2\pi i$.
- sinh and cosh are differentiable and $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$.

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Plan for tomorrow

Tomorrow we shall

- study second-order linear differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.

Section 4.1 and 4.3 in “Second-Order Equations” (pages xxxv-xlv and xlix-lv).

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Second-order linear differential equations

A *second-order linear differential equation* is a differential equation with can be written on the form

$$y'' + p(t)y' + q(t)y = g(t).$$

Such an equation is *homogeneous* if $g(t) = 0$.