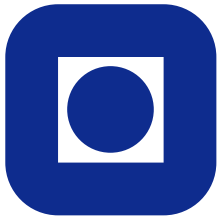

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# TMA4115 - Calculus 3

## Lecture 28, April 25

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Spring 2013

# Yesterday's lecture

Yesterday we looked at

- *least-squares problems*,
- applications to linear models.

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# Today's lecture

Today we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

# Symmetric matrices

- A *symmetric* matrix is a matrix $A$ such that $A^T = A$.
- A symmetric matrix is necessarily square.
- We will in this lecture see that every symmetric matrix is orthogonally diagonalizable, that is, if $A$ is symmetric, then $A = PDP^{-1}$ where $P$ is an orthogonal matrix and $D$ is a diagonal matrix.

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# Examples of symmetric matrices

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -3 \\ 1 & 2 & 4 \\ -3 & 4 & 7 \end{bmatrix}$$

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# Orthogonal matrices

Recall that an *orthogonal matrix* is a square matrix whose columns form an orthonormal set.

## Theorem

Let $P$ be an $n \times n$ matrix. Then the following statements are logically equivalent:

1. $P$ is an orthogonal matrix.
2. The columns of $P$ form an orthonormal basis for $\mathbb{R}^n$.
3. $P^T P = I_n$.
4. $P$ is invertible and $P^{-1} = P^T$.
5. $PP^T = I_n$.
6. The rows of $P$ form an orthonormal basis for $\mathbb{R}^n$.
7. $P^T$ is an orthogonal matrix.

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# Orthogonally diagonalization

An $n \times n$ matrix $A$ is *orthogonally diagonalizable* if there is an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

**Theorem 2**

An $n \times n$ matrix $A$ is orthogonally diagonalizable if and only if $A$ is symmetric.

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# Orthogonal eigenvectors

## Theorem 1

If $A$ is a symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.

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# The spectral theorem for symmetric matrices

## Theorem 3

An $n \times n$ symmetric matrix $A$ has the following properties:

1. $A$ has $n$ real eigenvalues, counting multiplicities.
2. The dimensions of the eigenspace for each eigenvalue $\lambda$ equals the multiplicity of $\lambda$.
3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

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# Spectral decomposition

Suppose $A = PDP^{-1}$ where $P = [\mathbf{u}_1 \dots \mathbf{u}_n]$ is an orthogonal matrix and $D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$.

If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$ and $A\mathbf{x} = \lambda_1(\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + \lambda_n(\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$.

We furthermore have that

$$A = \lambda \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

This is called a *spectral decomposition* of $A$.

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# Quadratic forms

- A *quadratic form* on $\mathbb{R}^n$ is a function $Q$ defined on $\mathbb{R}^n$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric $n \times n$ matrix $A$.
- The matrix $A$ is called the *matrix of* $Q$.
- Quadratic forms occupy a central place in various branches of mathematics, including
  - number theory,
  - linear algebra,
  - group theory,
  - differential geometry,
  - differential topology,
  - Lie theory.

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# The principal axes theorem

## Theorem 4

Let $A$ be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product term.

The above theorem can be used to classify conic sections.

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# Classifying quadratic forms

## Definition

A quadratic form $Q$ is said to be:

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- *indefinite* if $Q(\mathbf{x})$ assumes both positive and negative values.

- If $Q$ is a positive definite quadratic form on $\mathbb{R}^2$, then set of points $(x_1, x_2)$ satisfying $Q(x_1, x_2) = 1$ forms an ellipse.
- If $Q$ is an indefinite quadratic form on $\mathbb{R}^2$, then set of points $(x_1, x_2)$ satisfying $Q(x_1, x_2) = 1$ forms a hyperbola.

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# Quadratic forms and eigenvalues

## Theorem 5

Let $A$ be an $n \times n$ symmetric matrix, and let $Q$ be the quadratic form $x \mapsto \mathbf{x}^t A\mathbf{x}$.

1. $Q$ is positive definite if and only if the eigenvalues of $A$ are all positive.
2. $Q$ is negative definite if and only if the eigenvalues of $A$ are all negative.
3. $Q$ is indefinite if and only if $A$ has both positive and negative eigenvalues.

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