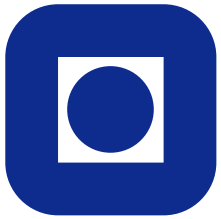

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TMA4115 - Calculus 3

Lecture 27, April 24

Toke Meier Carlsen
Norwegian University of Science and Technology

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Review of last week's lecture

Last week we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- *orthogonal* matrices,
- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

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Today's lecture

Today we shall look at

- *least-squares problems*,
- applications to linear models.

Least-squares problems

- Let A be an $m \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$.
- A *least-square solution* of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

  for all $\mathbf{x}$ in $\mathbb{R}^n$.
- The number $\|\mathbf{b} - A\hat{\mathbf{x}}\|$ is called the *least-squares error* of $A\mathbf{x} = \mathbf{b}$.

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Solution of the general least-squares problem

Theorem 13

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$.

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Uniqueness of least-squares solutions

Theorem 14

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

1. The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
2. The columns of A are linearly independent.
3. The matrix $A^T A$ is invertible.

When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$.

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QR-factorization and least-squares solutions

Theorem 15

Let A be an $m \times n$ matrix with linearly independent columns, let $A = QR$ be a QR-factorization of A and let $\mathbf{b}$ be in $\mathbb{R}^m$. Then $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$ is the unique least-squares solution of $A\mathbf{x} = \mathbf{b}$.

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Least-squares lines

- Suppose we are given a finite number of points $(x_1, y_1), \ldots, (x_n, y_n)$.
- We want to determine the line $y = \beta_0 + \beta_1 x$ which is as "close" to the points as possible.
- For each point (x_j, y_j) we call
  - y_j for the *observed* value of y,
  - $\beta_0 + \beta_1 x_j$ for the *predicted* value of y,
  - $y_j - (\beta_0 + \beta_1 x_j)$ for the *residual*.
- The *least-squares line* is the line $y = \beta_0 + \beta_1 x$ that minimizes the sum of the squares of the residuals.

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Finding the least-squares line

Finding the least-squares line is equivalent to computing the least-squares solution of the equation $X\beta = \mathbf{y}$ where

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

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The general linear model

- Any equation of the form $X\beta = \mathbf{y} + \epsilon$ where X and $\mathbf{y}$ are given, is referred to as a *linear model*.
- The goal is to minimize the length of the *residual vector* ϵ.
- This amounts to finding a least-squares solution of $X\beta = \mathbf{y}$.

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Tomorrow's lecture

Tomorrow we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

Sections 7.1–7.2 in "Linear Algebras and Its Applications" (pages 393–407).

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