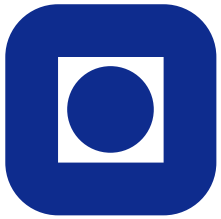

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TMA4115 - Calculus 3

Lecture 26, April 18

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Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- the *orthogonal complement* of a subspace,
- *orthogonal bases* and *orthonormal bases*,
- *orthogonal* matrices.

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Today's lecture

Today we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

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Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.
$W^\perp$ is a subspace of $\mathbb{R}^n$.

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The orthogonal complements of Row(A) and Col(A)

Theorem 3

Let A be an $m \times n$ matrix. Then

1. $(\text{Row}(A))^{\perp} = \text{Nul}(A)$.
2. $(\text{Col}(A))^{\perp} = \text{Nul}(A^T)$.

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Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Theorem 5

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in W. Then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

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Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if
$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.
- If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and $\mathbf{y}$ is in W, then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \mathbf{y} \cdot \mathbf{v}_j$.

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Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
2. $\|U\mathbf{x}\| = \|\mathbf{x}\|$.
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

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Orthogonal matrices

A square matrix with **orthonormal** columns is called an *orthogonal matrix*.
A square matrix U is orthogonal if and only if U is invertible and $U^{-1} = U^T$.

Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).
- If $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$, then $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y} = \text{proj}_L \mathbf{y} + \mathbf{z}$.
- The vector $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$ is called the component of $\mathbf{y}$ orthogonal to L (or to $\mathbf{u}$).
- $\|z\|$ is called the *distance from* $\mathbf{y}$ *to* L.

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Example

Let $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let us compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

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The orthogonal decomposition theorem

Theorem 8

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\mathbf{y}$ can be written uniquely in the form $\mathbf{y} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.
2. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W, then $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ and $\mathbf{z} = \mathbf{y} - \mathbf{w}$.

The vector $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ is called the *orthogonal projection of* $\mathbf{y}$ *onto* W and is denoted by $\text{proj}_W \mathbf{y}$.

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The best approximation theorem

Theorem 9

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be in $\mathbb{R}^n$, and let $\mathbf{w} = \text{proj}_W \mathbf{y}$.
Then $\mathbf{w}$ is the closest point in W to $\mathbf{y}$ in the sense that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}$ in W distinct from $\mathbf{w}$.

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Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us find the closest point in W to $\mathbf{y}$.

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Orthogonal projections onto orthonormal bases

Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$.
2. If $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$, then $\text{proj}_W \mathbf{y} = UU^T\mathbf{y}$.

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Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and

$W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

Let us find an orthogonal basis for W.

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An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
3. If appropriate, scale $\mathbf{v}_2$ to simplify later calculations.
4. Let $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$, let $W_3 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and scale $\mathbf{v}_3$ to simplify later calculations (if appropriate).

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An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

5. Continue like this and produce vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ where, for $1 < k \leq p$, $\mathbf{v}_k$ is an appropriate multiple of $\mathbf{x}_k - \text{proj}_{\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k - \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\mathbf{v}_{k-1}$.
6. Then $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is an orthogonal basis for W.

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The Gram-Schmidt process

Theorem 11

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a linearly independent subset of $\mathbb{R}^n$. Let

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_k - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set, and
$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $1 \leq k \leq p$.

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The QR factorization

Theorem 12

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where

- Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col(A),
- R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

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Example

Let us find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Problem 6 from June 2010

Let A be the following matrix; find a basis for each of the spaces Nul(A), Col(A), $(\text{Col}(A))^{\perp}$, and Row(A).

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix}$$

Find the orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to Col(A).

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Problem 5 from June 2011

Let V be the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

and let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}.$$

Find the nearest point in V to $\mathbf{b}$ (that is, the orthogonal projection of $\mathbf{b}$ on to V).

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Problem 5 from December 2010

Let $V \subseteq \mathbb{R}^4$ be the solution space of the linear system

$$x + y - z + w = 0$$
$$x + 2y - 2z + w = 0$$

1. Find an orthogonal basis for V.
2. Find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ on to V.
3. Find an orthogonal basis for $\mathbb{R}^4$ in which the first two first basis vectors are the once we found in (1).

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Plan for next week

Wednesday we shall look at

- *least-squares problems*,
- applications to linear models.

Sections 6.5–6.6 in “Linear Algebras and Its Applications” (pages 360–375).
Thursday we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

Sections 7.1–7.2 in “Linear Algebras and Its Applications” (pages 393–407).

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