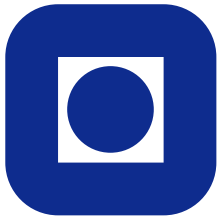

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TMA4115 - Calculus 3

Lecture 25, April 17

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Review of last week's lecture

Last week we studied

- real matrices with complex eigenvalues,
- systems of first order differential equations.

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Today's lecture

Today we shall introduce and study

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- *orthogonal* matrices.

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The inner product

- If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$, then the *inner product* (or *dot product*) of $\mathbf{u}$ and $\mathbf{v}$ are defined by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$.
- If we regard $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

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Properties of the inner product

Theorem 1

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

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The length of a vector

- If $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is a vector in $\mathbb{R}^n$, then the *length* (or *norm*) of $\mathbf{v}$ is the nonnegative number $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$.
- A *unit vector* is a vector whose length is 1.
- If $\mathbf{v}$ is in $\mathbb{R}^n$ and c is a scalar, then $\|c\mathbf{v}\| = |c|\|v\|$.
- If $\mathbf{v}$ is a nonzero vector then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector.
- The process of creating $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ from $\mathbf{v}$ is called *normalizing* $\mathbf{v}$.
- If $\mathbf{u} = c\mathbf{v}$ where $c > 0$, then we say that $\mathbf{u}$ is *in the same direction* as $\mathbf{v}$.

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Distance in $\mathbb{R}^n$

- If **u** and **v** are vectors in $\mathbb{R}^n$, then the *distance between* ***u*** *and* ***v***, written as dist(u, v), is the length of $\mathbf{u} - \mathbf{v}$.
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\| = \text{dist}(\mathbf{v}, \mathbf{u})$.

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Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$

If **u** and **v** are nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$$

where θ is the angle between the two line segments from the origin to the points identified with **u** and **v**.

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Orthogonal vectors and the Pythagorean theorem

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are *orthogonal* to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 2

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

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Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$.

- If a vector $\mathbf{z}$ is orthogonal to every vector in W, then $\mathbf{z}$ is said to be *orthogonal to* W.
- The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.
- If $W = \operatorname{Span} S$, then $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every $\mathbf{v}$ in S.
- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- $(W^\perp)^\perp = W$.

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The orthogonal complements of Row(A) and Col(A)

Theorem 3

Let A be an $m \times n$ matrix. Then

1. $(\text{Row}(A))^{\perp} = \text{Nul}(A)$.
2. $(\text{Col}(A))^{\perp} = \text{Nul}(A^T)$.

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Example

Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$. Let us determine $W^{\perp}$.

Orthogonal sets

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

Theorem 4

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence a basis for Span S.

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Orthogonal bases

- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Theorem 5

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in W. Then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

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Orthonormal sets

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if
$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.
- If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and $\mathbf{y}$ is in W, then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \mathbf{y} \cdot \mathbf{u}_j$.

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Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Matrices with orthonormal columns

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
2. $\|U\mathbf{x}\| = \|\mathbf{x}\|$.
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

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Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).
- If $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$, then $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y} = \text{proj}_L \mathbf{y} + \mathbf{z}$.
- The vector $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$ is called the component of $\mathbf{y}$ orthogonal to L (or to $\mathbf{u}$).
- $\|z\|$ is called the *distance from* $\mathbf{y}$ *to* L.

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Example

Let $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let us compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

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Tomorrow's lecture

Tomorrow we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

Sections 6.3–6.4 in "Linear Algebras and Its Applications" (pages 347-360).

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