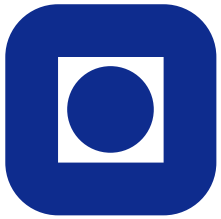

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TMA4115 - Calculus 3

Lecture 24, April 11

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Yesterday's lecture

Yesterday we studied

- real matrices with complex eigenvalues,
- systems of first order differential equations.

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Today's lecture

Today we shall continue studying systems of first order differential equations.

Systems of first order differential equations

A *system of first order differential equations with constant coefficients* is a system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

where $x_1, \ldots, x_n$ are differentiable functions of t with derivatives $x_1', \ldots, x_n'$, and the a_{ij} are constants.

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Systems of first order differential equations

If we let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then we write the system as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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The general solution to a system of first order differential equations

If an $n \times n$ matrix A has n linear independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$$

where $c_1, \ldots, c_n$ are constants.

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Attractors, repellers and saddle points

Let $\mathbf{x}'(t) = A\mathbf{x}(t)$ be a system of first order differential equations.

- The origin is called an *attractor* (or a *sink*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to \infty$.
- The origin is called a *repeller* (or a *source*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to -\infty$.
- The origin is called a *saddle point* if there is a solution $\mathbf{x}_1$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to \infty$ and a solution $\mathbf{x}_2$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to -\infty$.

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Complex eigenvalues

Let A be a real 2×2 matrix with a complex eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Let

$$\mathbf{y}_1(t) = \big((\mathrm{Re}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t) - (\mathrm{Im}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

$$\mathbf{y}_2(t) = \big((\mathrm{Re}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t) + (\mathrm{Im}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

Then $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}(t)$ is the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Writing second-order differential equations as systems of first order differential equations

- Let $y'' + ay' + by = 0$ be a homogeneous second-order differential equation with constant coefficients.
- Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$.
- Then the equation $y'' + ay' + by = 0$ is equivalent to the system

$$x_1' = x_2$$
$$x_2' = -bx_1 - ax_2$$

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Example

Let us find the solution to the initial values problem

$$y'' - y' - 6y = 0,\ y(0) = 3,\ y'(0) = 4.$$

Problem 6 from August 2011

Two tanks contain salt water. Tank 1 holds 200 litres and tank 2 holds 100 litres. Pure water enters tank 1 at a rate of 4 litres per second, salt water flows from tank 1 to tank 2 at the rate of 9 litres per seconds, salt water flows from tank 2 to tank 1 at the rate of 5 litres per second, and salt water flows from tank 2 and out of the system at a rate of 4 litres per second. The tanks are stirred so that the salt in each is evenly distributed.

1. Find a system of differential equations which describes the amount of salt $y_1(t)$ and $y_2(t)$ in, respectively, tank 1 and tank 2.
2. Determine $y_1(t)$ and $y_2(t)$ given that at the start ($t = 0$), tank 1 contained 100 grammes of salt and tank 2 contained 200 grammes of salt.

Problem 8 from June 2012

Let $A = \begin{bmatrix} -2 & -5 \\ 5 & -2 \end{bmatrix}$.

1. Find the complex eigenvalues of A and the corresponding eigenvectors in $\mathbb{C}^2$.
2. Find the solutions of the system of differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$ satisfying $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   The answer should be written in the form $\mathbf{y}(t) = e^{\lambda t} \begin{bmatrix} a\cos(\omega t) + b\sin(\omega t) \\ c\cos(\omega t) + d\sin(\omega t) \end{bmatrix}$.

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Plan for next week

Wednesday we shall introduce and study

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$.

Sections 6.1–6.2 in “Linear Algebras and Its Applications” (pages 329-346).

Thursday we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

Sections 6.3–6.4 in “Linear Algebras and Its Applications” (pages 347-360).

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