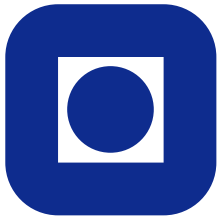

NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 23, April 10

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of last week's lecture

Last week we introduce and studied

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix,
- *diagonal* matrices,
- how to *diagonalizable* a matrix.

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall study

- real matrices with complex eigenvalues,
- systems of first order differential equations.

NTNU
Norwegian University of
Science and Technology

Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. Recall that:

- A scalar λ is an eigenvalue of A if and only if λ is a zero of the characteristic polynomial $\det(A - \lambda I_n)$ of A.
- If λ is an eigenvalue of A, then an eigenvector of A corresponding to λ is a nontrivial solution to the equation $A\mathbf{x} = \lambda\mathbf{x}$.

NTNU
Norwegian University of
Science and Technology

Complex eigenvalues and eigenvectors

- A real matrix A might have a complex eigenvalue λ in which case the corresponding eigenvectors are also complex.
- If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\mathbf{v}}$.

NTNU
Norwegian University of
Science and Technology

Rotation and scaling in $\mathbb{R}^2$

Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are real numbers which are not both zero, and let $r = |a + ib| = \sqrt{a^2 + b^2}$ and $\theta = \text{Arg}(a + ib)$.

Then $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation by $\text{Arg}\,(a + bi)$ followed by a scaling by $|a + bi|$.

Real 2×2 matrices with complex eigenvalues

Theorem 9

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}$$

where $P = [\mathrm{Re}(\mathbf{v}) \ \ \mathrm{Im}(\mathbf{v})]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

NTNU
Norwegian University of
Science and Technology

Problem 6 from the exam from December 2011

Let $A = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

1. Show that $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is a (complex) eigenvector of A.
2. Find the complex eigenvalues and complex eigenvectors of A.

NTNU
Norwegian University of Science and Technology

Systems of first order differential equations

A *system of first order differential equations with constant coefficients* is a system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

where $x_1, \ldots, x_n$ are differentiable functions of t with derivatives $x_1', \ldots, x_n'$, and the a_{ij} are constants.

NTNU
Norwegian University of Science and Technology

Systems of first order differential equations

If we let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then we write the system as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

NTNU
Norwegian University of
Science and Technology

Systems of first order differential equations

- If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are solutions to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, then any linear combination $c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$ of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is also a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$.
- The solution set of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is an n-dimensional vector space, so if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$, then any solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

NTNU
Norwegian University of
Science and Technology

Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

Attractors, repellers and saddle points

Let $\mathbf{x}'(t) = A\mathbf{x}(t)$ be a system of first order differential equations.

- The origin is called an *attractor* (or a *sink*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to \infty$.
- The origin is called a *repeller* (or a *source*) if every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ converges to $\mathbf{0}$ as $t \to -\infty$.
- The origin is called a *saddle point* if there is a solution $\mathbf{x}_1$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to \infty$ and a solution $\mathbf{x}_2$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ that converges to $\mathbf{0}$ as $t \to -\infty$.

NTNU
Norwegian University of Science and Technology

Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, where $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$, and determine if the origin is an attractor, repeller, or a saddle point.

The general solution to a system of first order differential equations

If an $n \times n$ matrix A has n linear independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_2 e^{\lambda_n t}$$

where $c_1, \ldots, c_n$ are constants.

NTNU
Norwegian University of
Science and Technology

Decoupling a dynamical system

- Consider a system $\mathbf{x}'(t) = A\mathbf{x}(t)$ of first order differential equations where A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.
- Let $P = [\mathbf{v}_1 \ \ldots \mathbf{v}_n]$ and let D be the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Then $A = PDP^{-1}$.
- If we let $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then $\mathbf{x}'(t) = A\mathbf{x}(t)$ is equivalent to $\mathbf{y}'(t) = D\mathbf{y}(t)$.
- It follows that the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$.

NTNU
Norwegian University of
Science and Technology

Example

Let us find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$.

Complex eigenvalues

Let A be a real 2×2 matrix with a complex eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Let

$$\mathbf{y}_1(t) = \big((\mathrm{Re}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t) - (\mathrm{Im}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

$$\mathbf{y}_2(t) = \big((\mathrm{Re}(\mathbf{v}))\sin(\mathrm{Im}(\lambda)t) + (\mathrm{Im}(\mathbf{v}))\cos(\mathrm{Im}(\lambda)t)\big)e^{\mathrm{Re}(\lambda)t}$$

Then $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}(t)$ is the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

NTNU
Norwegian University of
Science and Technology

Problem 6 from August 2011

Two tanks contain salt water. Tank 1 holds 200 litres and tank 2 holds 100 litres. Pure water enters tank 1 at a rate of 4 litres per second, salt water flows from tank 1 to tank 2 at the rate of 9 litres per seconds, salt water flows from tank 2 to tank 1 at the rate of 5 litres per second, and salt water flows from tank 2 and out of the system at a rate of 4 litres per second. The tanks are stirred so that the salt in each is evenly distributed.

1. Find a system of differential equations which describes the salt concentrations $y_1(t)$ and $y_2(t)$ of, respectively, tank 1 and tank 2.
2. Determine $y_1(t)$ and $y_2(t)$ given that at the start ($t = 0$), tank 1 contained 100 grammes of salt and tank 2 contained 200 grammes of salt.

Problem 8 from June 2012

Let $A = \begin{bmatrix} -2 & -5 \\ 5 & -2 \end{bmatrix}$.

1. Find the complex eigenvalues of A and the corresponding eigenvectors in $\mathbb{C}^2$.
2. Find the solutions of the system of differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$ satisfying $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   The answer should be written in the form $\mathbf{y}(t) = e^{\lambda t} \begin{bmatrix} a\cos(\omega t) + b\sin(\omega t) \\ c\cos(\omega t) + d\sin(\omega t) \end{bmatrix}$.

NTNU
Norwegian University of
Science and Technology

Tomorrow's lecture

Tomorrow we shall continue studying systems of first order differential equations. Section 5.7 in "Linear Algebras and Its Applications" (pages 295-301, and 311-319), and Section 4.2 in "Second-Order Equations" (pages xlv-xlix).

NTNU
Norwegian University of
Science and Technology