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TMA4115 - Calculus 3

Lecture 20, March 21

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Yesterday’s lecture

Yesterday we introduced and studied

- the *dimension* of a vector space,
- the *rank* of a matrix.

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Today's lecture

Today we shall look at

- the *rank* of a matrix,
- *Markov chains*.

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The dimension of a vector space

Theorem 10

If a vector space has a basic of n vectors, then every basis of V must consist of exactly n vectors.

- If V is spanned by a finite set, then V is said to be *finite-dimensional*.
- If V is a finite dimensional vector space, then the *dimension* of V, written as $\dim(V)$, is the number of vectors in a basis for V.
- The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.
- If V is not spanned by a finite set, then V is said to be *infinite-dimensional*.

Why is the dimension of a vector space useful?

If V is a vector space of dimension n, then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an *isomorphism* between V to $\mathbb{R}^n$ for any basis $\mathcal{B}$ of V.

Theorem 12

Let V be a p-dimensional vector space, where $p \geq 1$.

1. Any linearly independent set of exactly p elements in V is automatically a basis for V
2. Any set of exactly p elements that spans V is automatically a basis for V.

It follows that if V is a finite dimensional vector space, H is a subspace of V and $\dim(H) = \dim(V)$, then $H = V$.

The dimension of the column space of a matrix

Recall that if $A = [\mathbf{a}_1\ \mathbf{a}_2 \dots \mathbf{a}_n]$ is an $m \times n$ matrix, then $\mathrm{Col}(A) = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.

Recall also that the pivot columns of A form a basis for $\mathrm{Col}(A)$.

It follows that the dimension of $\mathrm{Col}(A)$ is the number of pivot positions in A.

The dimension of the null space of a matrix

Recall that if A is an $m \times n$ matrix, then
$\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
Recall also that we can find a basis for Nul(A) by

1. row reducing A to its reduced echelon form,
2. writing the solutions to the equation $A\mathbf{x} = \mathbf{0}$ as a linear combinations of vectors using the free variables as parameters,
3. forming a basis for Nul(A) by taken the set of the vectors we use to write the solutions to the equation $A\mathbf{x} = \mathbf{0}$.

It follows that the dimension of Nul(A) is the number of nonpivot columns of A.

Problem 5 from June 2012

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $T\left(\begin{bmatrix} x \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} 2x + y + z \\ -x + 3y + z \\ 2x - z \\ y + 4z \end{bmatrix}$.

a) Find a matrix A such that $T\left(\begin{bmatrix} x \\ y \\ x \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ x \end{bmatrix}$.

b) Find dim(Nul(A)) and a basis for Col(A). Is T one-to-one (injective)? Is T onto (surjective)?

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The dimension of the row space of a matrix

Recall that the *row space* of a matrix A is set of all linear combination of the row vectors of A. Recall also that we have the following theorem:

Theorem 13

Let A and B be matrices.

1. If A and B are row equivalent, then $\text{Row}(A) = \text{Row}(B)$.
2. If B is in echelon form, then the nonzero rows of B form a basis for $\text{Row}(B)$.

It follows that the dimension of $\text{Row}(A)$ is the number of pivot positions in A.

The rank of a matrix

Let A be a matrix. Then the *rank* of A, written $\text{rank}(A)$, is the dimension of the column space $\text{Col}(A)$ of A.

Theorem 14

Let A be an $m \times n$ matrix. Then the following numbers are equal:

- $\text{rank}(A)$
- $\dim(\text{Col}(A))$
- $\dim(\text{Row}(A))$
- $n - \dim(\text{Nul}(A))$
- The number of pivot positions in A.

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Example

Let A be a nonzero 3×3 matrix. Suppose that

$$A \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{0}.$$

What is $\dim(\text{Row}(A))$?

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The rank of a matrix and solutions to linear system

Consider a linear system in m variables, and let A be the coefficient matrix of the system and B the augmented matrix of the system.

1. Then $\operatorname{rank}(A) \leq \operatorname{rank}(B)$.
2. The system is consistent if and only if $\operatorname{rank}(A) = \operatorname{rank}(B)$.
3. If the system is consistent, then it has a unique solution if and only if $\operatorname{rank}(A) = m$.
4. If the system is consistent, then its general solution has $m - \operatorname{rank}(A)$ free variables.

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Problem 4 from the exam from August 2011

Let $A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$.

1. Find a basis for the null space, Nul(A), and a basis for the row space, Row(A).
2. Find a basis for the column space, Col(A). What is rank(A).

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The invertible matrix theorem

Theorem

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is invertible.

1. The columns of A form a basis of $\mathbb{R}^n$.
2. $\mathrm{Col}(A) = \mathbb{R}^n$.
3. $\dim(\mathrm{Col}(A)) = n$.
4. $\mathrm{rank}(A) = n$.
5. $\mathrm{Nul}(A) = \{\mathbf{0}\}$.
6. $\dim(\mathrm{Nul}(A)) = 0$.
7. $\dim(\mathrm{Row}(A)) = n$.
8. The rows of A form a basis of $\mathbb{R}^n$.
9. $\mathrm{Row}(A) = \mathbb{R}^n$.

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Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city. In 2000, there were 600,000 residents in the city and 400,000 in the suburbs. Let us look at how the populations of the city and of the surrounding suburbs change through the years.

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Markov chains

- A *probability vector* is a vector with nonnegative entries that add up to 1.
- A *stochastic matrix* is a square matrix whose columns are probability vectors.
- A *Markov chain* is a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots$ of probability vectors together with a stochastic matrix P such that $\mathbf{x}_1 = P\mathbf{x}_0$, $\mathbf{x}_2 = P\mathbf{x}_1$, $\mathbf{x}_3 = P\mathbf{x}_2$, $\dots$.
- A *steady-state vector* (or an *equilibrium vector*) for a stochastic matrix P is a probability vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.

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Convergence of regular Markov chains

A stochastic matrix P is said to be *regular* if P^k contains only strict positive entries for some k.

Theorem 18

- If P is a regular stochastic matrix, then P has a unique steady-state vector q.
- If $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots,$ then the Markov chain $\{\mathbf{x}_k\}_{k\in\mathbb{N}_0}$ converges to $\mathbf{q}$ as $k \to \infty$.

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Applications of Markov chains

Markov chains is a useful tool that can be used to model discrete dynamical systems.
They are for example used in

- physics,
- chemistry,
- information theory,
- statistics,
- economics and finance,
- biology,
- social sciences.

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Example

A very simple weather model says that if it is raining one day, then there is a 60% chance that will also rain the next day, and that if it not raining, then there is a 30% chance that will rain the next day. According to this model, what is the probability that it will rain on a given day?

Plan the week after the Easter break

Wednesday we shall introduce and study

- *eigenvectors*, *eigenvalues* and *eigenspaces* of square matrices,
- the *characteristic polynomial* of a square matrix.

Sections 5.1–5.2 in “Linear Algebras and Its Applications” (pages 265—281).

Thursday we shall look at *diagonalizable* matrices. Section 5.3 in “Linear Algebras and Its Applications” (pages 281—288).

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