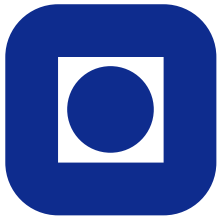


NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 2, Jan 17

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Course web page

Information about the course can be found at
`http://wiki.math.ntnu.no/tma4115/2013v`.

NTNU
Norwegian University of
Science and Technology

Review of yesterday's lecture

Yesterday we introduced *complex numbers* and studied

- the *real part*, the *imaginary part*, the *absolute value* (or *modulus*), and the *argument* of a complex number,
- addition and multiplication of complex numbers,
- and *complex conjugation*.

NTNU
Norwegian University of
Science and Technology

Today's lecture

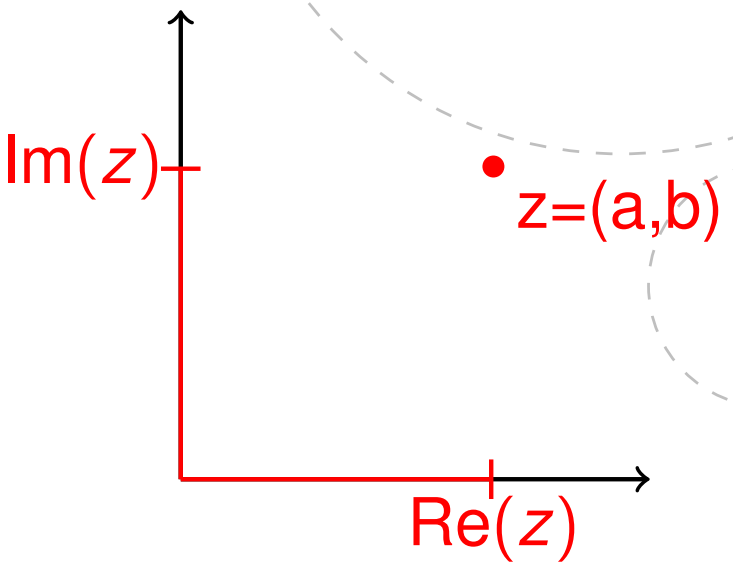
Today we will study

- *polar representation* of complex numbers,
- *de Moivre's Theorem*,
- how complex numbers can be used to derive trigonometric identities,
- roots of complex numbers.

NTNU
Norwegian University of
Science and Technology

Complex numbers

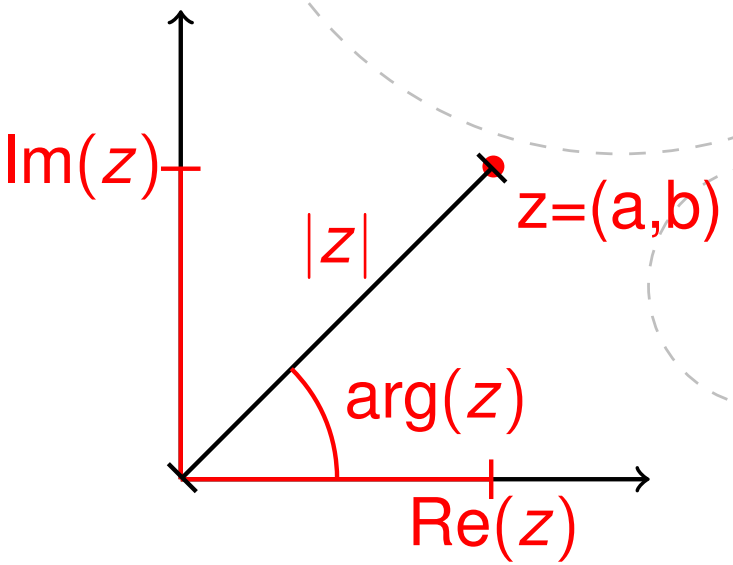
- A complex number is a number which can be written as $a + ib$ where a and b are real numbers and i satisfies $i^2 = -1$.
- A complex number $z = a + ib$ can be represented as the point (a, b) in the plane.
- If $z = a + ib$, then a is called the *real part* of z and is denoted by $\mathrm{Re}(z)$.
- b is called the *imaginary part* of z and is denoted by $\mathrm{Im}(z)$.



NTNU
Norwegian University of Science and Technology

Complex numbers

- If $z = a + ib$, then the length $\sqrt{a^2 + b^2}$ of the line from $(0, 0)$ to (a, b) is called the *modulus* or the *absolute value* of z and is denoted by $|z|$.
- The angle between the line through $(0, 0)$ and (a, b) and the positive part of the real axis is called the *argument* of z and is denoted by $\arg(z)$.



NTNU
Norwegian University of
Science and Technology

Complex numbers

- $\arg(z)$ is not unique. If $\theta = \arg(z)$, then also $\theta + 2\pi = \arg(z)$. If we want to be precise, then $\arg(z)$ is really the set of all angles θ which satisfies that if we rotate the positive part of the real axis by θ, then it lands on the line through $(0, 0)$ and (a, b).
- The unique value of $\arg(z)$ in the interval $(-\pi, \pi]$ is called the *principal argument* of z and is denoted by $\mathrm{Arg}(z)$.
- Notice that $\arg(z)$ and $\mathrm{Arg}(z)$ are not defined if $z = (0, 0)$.

NTNU
Norwegian University of
Science and Technology

Addition of complex numbers

- If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.
- $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$.
- $z_1 + z_2 = z_2 + z_1$.
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
- $|z_1 + z_2| \leq |z_1| + |z_2|$.

NTNU
Norwegian University of
Science and Technology

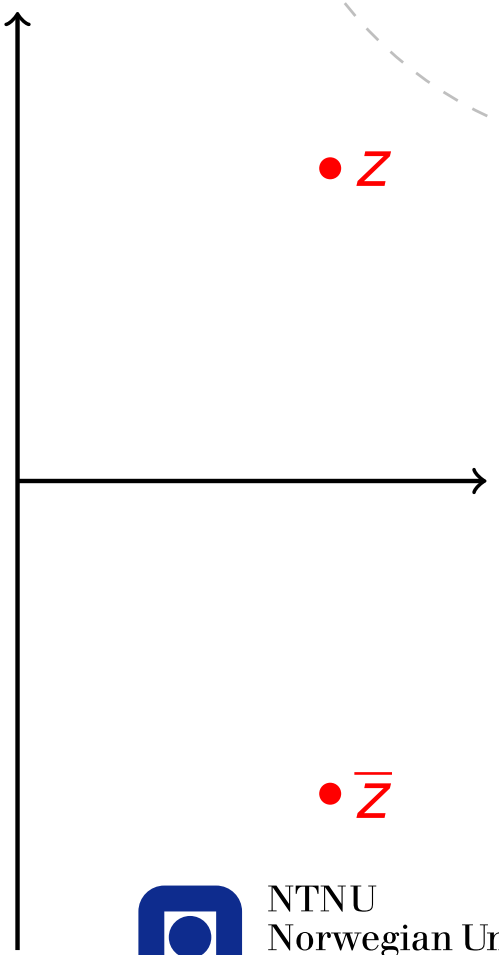
Multiplication of complex numbers

- If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 z_2 = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $|z_1 z_2| = |z_1||z_2|$.
- $z_1 z_2 = z_2 z_1$.
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

NTNU
Norwegian University of
Science and Technology

Complex conjugation

- When $z = a + bi$ is a complex number, then the number $a - bi$ is called the *conjugate* of z and is denoted by $\overline{z}$.
- $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$.
- We get $\overline{z}$ by reflecting z in the real line.
- $|\overline{z}| = |z|$ and $\arg(\overline{z}) = -\arg(z)$.



NTNU
Norwegian University of
Science and Technology

Complex conjugation

- $z = \overline{z}$ if and only if $\mathrm{Im}(z) = 0$.
- $z = -\overline{z}$ if and only if $\mathrm{Re}(z) = 0$.
- $\overline{z + w} = \overline{z} + \overline{w}$.
- $\overline{zw} = \overline{z}\,\overline{w}$.
- $z\overline{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 = |z|^2$.

NTNU
Norwegian University of
Science and Technology

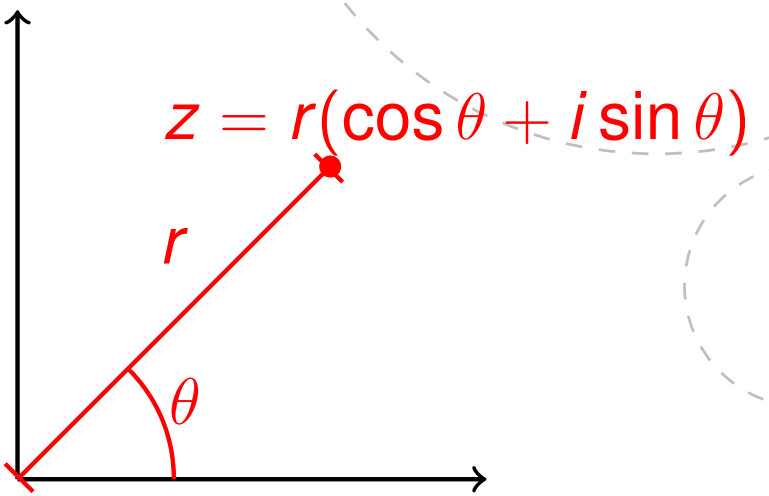
Division of complex numbers

- If z and w are complex numbers and $w \neq 0$, then
$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}.$$
- $$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$
- $$\arg\left(\frac{z}{w}\right) = \frac{\arg(z)}{\arg(w)} \text{ and } \left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

NTNU
Norwegian University of
Science and Technology

Polar representation

- Every complex number z can be written on the form $r(\cos\theta + i\sin\theta)$ where r and θ are real numbers and $r \geq 0$. This is called the *polar form* of z.
- Notice that if $z = r(\cos\theta + i\sin\theta)$, then $|z| = r$ and $\arg(z) = \theta$.



Multiplication and division using polar representations

Recall that

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $|z_1 z_2| = |z_1||z_2|$, and that
- $\arg\left(\frac{z}{w}\right) = \frac{\arg(z)}{\arg(w)}$ and $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

It follows that if $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

- $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ and
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ (provided $r_2 \neq 0$).

NTNU
Norwegian University of Science and Technology

Exercise 46, page xviii

Express each of the complex numbers $z = 3 + i\sqrt{3}$ and $w = -1 + i\sqrt{3}$ in polar form. Use these expressions to calculate zw and z/w.

Complex numbers and trigonometric identities

Polar representations of complex numbers can be used to derive trigonometric identities like

- $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$
- $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$

NTNU
Norwegian University of
Science and Technology

de Moivre's Theorem

- Recall that $\arg(zw) = \arg(z) + \arg(w)$ and $|zw| = |z||w|$.
- It follows that $(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
- We have in particular that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

- This formula is know as *de Moivre's formula*.

NTNU
Norwegian University of
Science and Technology

Complex numbers and trigonometric identities

Polar representations of complex numbers can be used to derive trigonometric identities like

- $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$
- $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$
- $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$
- $\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$

NTNU
Norwegian University of
Science and Technology

Roots of complex numbers

- If z is a complex number and n is a positive integer, then an *n'th root* of z is a complex number w satisfying $w^n = z$.
- Every complex number different from 0 has n different n'th roots.

Roots of complex numbers

If z is a complex number different from 0 and n is a positive integer, then the n'th roots of z are

$$w_k = |z|^{1/n}\left(\cos\left(\frac{\arg(z)+2\pi k}{n}\right)+i\sin\left(\frac{\arg(z)+2\pi k}{n}\right)\right)$$

where $k = 0, 1, \ldots, n-1$.

Roots of unity

- If n is a positive integer, then a n'th root of 1 is called a *n'th root of unity*.
- The n n'th roots of unity are

$$u_n = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.

Roots of complex numbers

If z is a complex number different from 0, n is a positive integer and w is an n'th root of z, then the other $n-1$ n'th roots of z are wu_1, wu_2, $\ldots$ wu_{n-1} where u_1, u_2, $\ldots$ u_{n-1} are the $n-1$ n'th roots of unity different from 1.

Next week's lectures

Wednesday we shall

- use complex numbers to solve polynomial equations,
- look at *the fundamental theorem of algebra*,
- introduce *the complex exponential function*,
- and study extensions of trigonometric functions to the complex numbers.

Thursday we shall

- study second-order differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.

NTNU
Norwegian University of
Science and Technology

Problem 1 from the exam from August 2012

Write all of the solutions of $z^3 = 1$ in the form $z = x + iy$.
Write the solutions of $z^3 = \frac{-3+i}{\sqrt{2}(2+i)}$ in the form $z = x + iy$ and draw the solutions in the complex plane.

NTNU
Norwegian University of Science and Technology

Problem 1 from the exam from June 2009

Find all complex numbers $z = x + iy$ which satisfy the equality $|z + 1 - i\sqrt{3}| = |z - 1 + i\sqrt{3}|$. Draw the solutions in a diagram.

NTNU
Norwegian University of
Science and Technology

Problem 1 from the exam from August 2011

Find all complex numbers z such that $\mathrm{Im}(-z+i) = (z+i)^2$. Draw the solutions on a diagram.

NTNU
Norwegian University of
Science and Technology

Problem 1 from the exam from December 2009

Find all of the solutions of

$$z^5 = \frac{16(2\sqrt{3} - 1 - i(2 + \sqrt{3}))}{2 - i}$$

and draw the solutions in the complex plane.

NTNU
Norwegian University of
Science and Technology