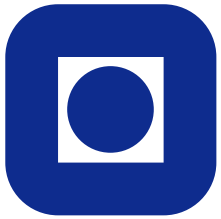

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TMA4115 - Calculus 3

Lecture 18, March 14

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Yesterday's lecture

Yesterday we introduced and studied

- abstract *vector spaces* and *subspaces*,
- *null spaces* and *column spaces* of matrices,
- *linear transformations* between abstract vector spaces,
- *kernels* and *ranges* of linear transformations,
- *linear independence* and *linear dependence* in abstract vector spaces.

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Today's lecture

Today we shall introduce and study

- *bases* of vector spaces,
- *coordinate systems* in vector spaces relative to bases.

Vector spaces

Definition

A *vector space* is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed on the next slide.

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1. The *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a *zero* vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The *scalar multiple* of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

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Subspaces

Definition

A *subspace* of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H.
3. If $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

Every subspace is a vector space. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

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Linear independence

Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V.

- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly independent* if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$ not all equal to zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

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Linear independence in $\mathbb{R}^n$

- Recall that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p]$ has a pivot position in every column.
- There is no similar method to determine if a indexed set of vectors in an arbitrary vector space is linearly independent.

Linear dependence

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors in a vector set V is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$ or some $\mathbf{v}_j$, with $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Bases

Definition

Let H be a subspace of a vector space V. An indexed set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of vectors in H is a *basis* for H if

1. $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly independent,
2. $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$.

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The standard basis for $\mathbb{R}^n$

- Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the columns of I_n.
- That is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.
- Since I_n is invertible, it follows from the inverse matrix theorem that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$ and is linearly independent.
- So $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.
- $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is called the *standard basis* for $\mathbb{R}^n$.

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Bases of $\mathbb{R}^n$

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$.
- If the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is invertible, then it follows from the inverse matrix theorem that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $\mathbb{R}^n$ and is linearly independent.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is invertible.

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The standard basis for $\mathbb{P}_n$

- $\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$.
- $\{1, t, t^2, \ldots, t^n\}$ is called the *standard basis* for $\mathbb{P}_n$.

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Bases for the solution set of homogeneous linear second-order differential equations

- Let $y'' + p(t)y' + q(t)y = 0$ be a homogenous second-order linear differential equation.
- Recall that the set of solutions to $y'' + p(t)y' + q(t)y = 0$ is the kernel of the linear transformation that maps a real-valued twice continuously differentiable function f to $f'' + p(t)f' + q(t)f$.
- Recall that if y_1 and y_2 are two linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then $\{y_1, y_2\}$ is called a fundamental set of solutions.

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Bases for the solution set of homogeneous linear second-order differential equations

- Recall also that if $\{y_1, y_2\}$ is a fundamental set of solutions, then every solution of $y'' + p(t)y' + q(t)y = 0$ is a linear combination of y_1 and y_2.
- So a fundamental set of solutions is a basis for the solutions set of $y'' + p(t)y' + q(t)y = 0$.

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The spanning set theorem

Theorem 5

Let V be a vector space, let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

1. If one of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then the set formed from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ by removing $\mathbf{v}_k$ still spans H.
2. If $H \neq \{\mathbf{0}\}$, then some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for H.

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The column space of a matrix

Let A be an $m \times n$ matrix.

1. The *column space* of A, written as $\text{Col}(A)$, is the set of all linear combinations of the columns of A.
2. If $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$, then $\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
3. $\text{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$.
4. $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

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Bases for the column space of a matrix

Theorem 6

Let A be an $m \times n$ matrix. Then the pivot columns of A form a basis for $\mathrm{Col}(A)$.

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The null space of a matrix

Let A be an $m \times n$ matrix.

1. The *null space* of A, written as $\text{Nul}(A)$, is the set of all solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
2. $\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
3. $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

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Finding a basis for Nul(A)

1. Row reduce A to its reduced echelon form.
2. Write the solutions to the equation $A\mathbf{x} = \mathbf{0}$ as a linear combinations of vectors using the free variables as parameters.
3. These vectors then form a basis for Nul(A).

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The unique representation theorem

Theorem 7

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.

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The coordinates of x relative to the basis $\mathcal{B}$

- Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and that $\mathbf{x}$ is in V. The unique set of scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$ is called the *coordinates of* $\boldsymbol{x}$ *relative to the basis* $\mathcal{B}$ (or the $\mathcal{B}$*-coordinates of* x).
- If $c_1, c_2, \ldots, c_n$ are the $\mathcal{B}$-coordinates of x, then the vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the *coordinate vector of* $\boldsymbol{x}$ *(relative to* $\mathcal{B}$*)*, or the $\mathcal{B}$*-coordinate vector of* $\boldsymbol{x}$.

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The coordinate mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ from V to $\mathbb{R}^n$ is called the *coordinate mapping determined by* $\mathcal{B}$.

Theorem 8

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one and onto linear transformation.

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Coordinates in $\mathbb{R}^n$

- Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$.
- Let $P_{\mathcal{B}} = [\mathbf{b}_1\ \mathbf{b}_2 \ldots \mathbf{b}_n]$. It follows from the inverse matrix theorem that P_B is invertible.
- If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$, from which it follows that $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$.

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Example

Suppose we have a large container that contains two chemical substances, substance A and substance B. Substance A contains 3 g sulphuric acid and 1 g acetic acid per 100 g, and substance B contains 5 g sulphuric acid and 2 g acetic acid per 100 g. How much of substance A and how much how substance B must we add or remove from the container in order to increase the amount of sulphuric acid in the container by 2 g and reduce the amount of acetic acid by 1 g?

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Plan for next week

Wednesday we shall introduce and study

- the *dimension* of a vector space,
- the *rank* of a matrix.

Sections 4.5–4.7 in “Linear Algebras and Its Applications” (pages 225–238).

Thursday we shall look at applications to *Markov chains*. Section 4.9 in “Linear Algebras and Its Applications” (pages 253–264).

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