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TMA4115 - Calculus 3

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Review of last week's lecture

Last week we looked at

- the *determinate* of a square matrix,
- *Cramer's rule*,
- a formula for the inverse of an invertible matrix,
- the relationship between areas, volumes and determinants.

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Today's lecture

Today we shall introduce and study

- abstract *vector spaces* and *subspaces*,
- *null spaces* and *column spaces* of matrices,
- *linear transformations* between abstract vector spaces,
- *kernels* and *ranges* of linear transformations,
- *linear independence* and *linear dependence* in abstract vector spaces.

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Vector spaces

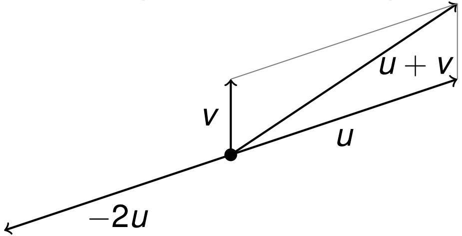
Definition

A *vector space* is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed on the next slide.

1. The *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a *zero* vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The *scalar multiple* of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces

1. Let n be a positive integer. Then $\mathbb{R}^n$ is a vector space.
2. The set of arrows in a fixed plane (or space), starting at one fixed point is a vector space.



3. Let m and n be positive integers. Then the set $M_{m \times n}$ of all $m \times n$ matrices is a vector space.

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Examples of vector spaces

3. Let $\mathbb{S}$ be the set of all doubly infinite sequences of numbers: $\{y_k\}_{k\in\mathbb{Z}}$. If $\{z_k\}_{k\in\mathbb{Z}}$ is another element of $\mathbb{S}$, then the sum $\{y_k\}_{k\in\mathbb{Z}} + \{z_k\}_{k\in\mathbb{Z}}$ is sequence $\{y_k + z_k\}_{k\in\mathbb{Z}}$, and if c is a scalar, then $c\{y_k\}_{k\in\mathbb{Z}}$ is the sequence $\{cy_k\}_{k\in\mathbb{Z}}$. Then $\mathbb{S}$ is a vector space.
4. Let I be an interval and let V be the space of all real-valued functions defined on I. If f, g are in V and c is a scalar, then $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for x in I, and cf is the function defined by $(cf)(x) = cf(x)$ for x in I. Then V is a vector space.

Subspaces

Definition

A *subspace* of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H.
3. If $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

Every subspace is a vector space. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

Examples of subspaces

Let V be the space of all real-valued functions defined on $\mathbb{R}$.

1. Let $\mathbb{P}$ be the set of all polynomials with real coefficients. Then $\mathbb{P}$ is a subspace of V.
2. For each $n \geq 0$, let $\mathbb{P}_n$ be the set of polynomials with real coefficients of degree at most n. Then $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ (and of V).
3. Let $C(\mathbb{R})$ be the set of all real-valued continuous functions defined on $\mathbb{R}$. Then $C(\mathbb{R})$ is a subspace of V.

The zero subspace

Let V be a vector space, and let $\mathbf{0}$ be the zero vector of V. The set consisting of only $\mathbf{0}$ is a subspace of V. This subspace is called the *zero subspace* and is written as $\{\mathbf{0}\}$.

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Subspaces spanned by sets

Let V be a vector space.

1. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in V and $c_1, c_2, \ldots, c_n$ are scalars, then the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
2. We let $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ denote the set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
3. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V.
4. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called the *subspace spanned* (or *generated*) by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
5. Given any subspace H of V, a *spanning* (or *generating*) set for H is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

The null space of a matrix

Let A be an $m \times n$ matrix.

1. The *null space* of A, written as Nul(A), is the set of all solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
2. $\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.
3. Nul(A) is a subspace of $\mathbb{R}^n$.

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The column space of a matrix

Let A be an $m \times n$ matrix.

1. The *column space* of A, written as $\mathrm{Col}(A)$, is the set of all linear combinations of the columns of A.
2. If $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$, then $\mathrm{Col}(A) = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
3. $\mathrm{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$.
4. $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

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Linear transformations

Definition

A *linear transformation* T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V,
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{u}$ in V all scalars c.

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Example

- Let V be the set of all real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$.
- Let $W = C[a, b]$ be the set of real-valued continuous functions on $[a, b]$.
- Then V and W are vector spaces.
- Let $D : V \to W$ be the transformation that maps f in V to its derivative f'.
- Then $D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$ and $D(cf) = (cf)' = cf' = cD(f)$, so D is a linear transformation.

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The kernel and range of a linear transformation

Let V and W be vector spaces and let T be a linear transformation from V to W.

1. The *kernel* (or *null space*) of T is the set $\{\mathbf{u} : \mathbf{u} \text{ is in } V \text{ and } T(\mathbf{v}) = \mathbf{0}\}$ of all $\mathbf{u}$ in V for which $T(\mathbf{u}) = \mathbf{0}$.
2. The *range* of T is the set $\{\mathbf{w} : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}$ of all vectors in W of the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V.
3. The kernel of T is a subspace of V.
4. The range of T is a subspace of W.
5. If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and A is the standard matrix of T, then the kernel of T is the null space of A, and the range of T is the column space of A.

Linear independence

Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an indexed set of vectors in V.

- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly independent* if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$ not all equal to zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

Linear independence in $\mathbb{R}^n$

- Recall that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if the matrix $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p]$ has a pivot position in every column.
- There is no similar method to determine if a indexed set of vectors in an arbitrary vector space is linearly independent.

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Linear dependence

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors in a vector set V is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$ or some $\mathbf{v}_j$, with $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

Tomorrow's lecture

Tomorrow we shall introduce and study

- *bases* of vector spaces,
- *coordinate systems* in vector spaces relative to bases.

Section 4.3–4.4 in "Linear Algebras and Its Applications" (pages 208–225).

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