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TMA4115 - Calculus 3

Lecture 16, March 7

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Yesterday’s lecture

Yesterday we introduce and study *determinants*.

Today's lecture

Today we shall

- look at *Cramer's rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\det(A) \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

Properties of determinants

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

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Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

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Cramer's rule

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

Theorem 7

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)} \text{ for } i = 1, 2, \ldots, n.$$

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Example

Let us find the values of the parameter s for which the system

$$\begin{aligned} 2sx_1 + x_2 &= 1 \\ 3sx_1 + 6sx_2 &= 2 \end{aligned}$$

has a unique solution, and then find this solution.

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An inverse formula

When A is an $n \times n$ matrix, then $\text{adj}(A)$ is the $n \times n$ matrix whose (i, j)-entry is $C_{ji} = (-1)^{i+j} \det(A_{ji})$.

Theorem 8

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

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Determinants as area or volume

Theorem 9

If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det(A)|$.
If A is a 3×3 matrix, then the area of the parallelepiped determined by the columns of A is $|\det(A)|$.

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Example

Let us find the area of the parallelogram whose vertices are $(-2, 0)$, $(-3, 3)$, $(2, -5)$ and $(1, -2)$.

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Areas and linear transformations

If T is a transformation and S is a set in the domain of T, then we let $T(S)$ denote the set of images of points in S.

Theorem 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a region in $\mathbb{R}^2$ with finite area, then area of $T(S) = |\det(A)|$(area of S).
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is a region in $\mathbb{R}^2$ with finite volume, then volume of $T(S) = |\det(A)|$(volume of S).

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Example

Let a and b be positive numbers. Let us find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

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Problem 5 from the exam from August 2011

Let $A = \begin{bmatrix} 1 & 2 & a \\ 3 & 7 & b \\ 2 & 9 & c \end{bmatrix}$.

1. Decide for which values of a, b and c, the matrix A is invertible.
2. Find values of a, b and c for which A^{-1} is an integer matrix.

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Problem 4 from August 2007

1. A square 3×3 matrix A is given by

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{bmatrix}$$

   For which real numbers a is the matrix A invertible?

2. Find A^{-1} when $a = 1$.

Plan for next week

Wednesday we shall introduce and study

- abstract *vector spaces* and *subspaces*,
- *null spaces*, *column spaces* and general *linear transformations*.

Sections 4.1–4.2 in “Linear Algebras and Its Applications” (pages 189–208).

Thursday we shall introduce and study

- *linear independence* and *bases* in general vector spaces,
- *coordinate systems* in vector spaces relative to bases.

Section 4.3–4.4 in “Linear Algebras and Its Applications” (pages 208–225).