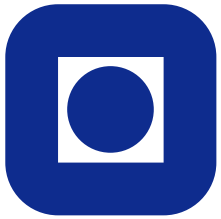

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TMA4115 - Calculus 3

Lecture 15, March 6

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Review of last week's lecture

Last week we looked at

- how to *add* and *multiply* matrices,
- *invertible* matrices and their *inverses*,
- the *the invertible matrix theorem*.

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Today's lecture

Today we shall introduce and study *determinants*.

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The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The number $ad - bc$ is called the *determinant* of A.

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Determinant

- The determinant is a value associated with a square matrix.
- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- The determinant can be used to give an explicit formula for the inverse of an invertible matrix.
- $\det(AB) = \det(A)\det(B)$.
- The absolute value of the determinant gives the scale factor by which area or volume is multiplied under the associated linear transformation.

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The definition of the determinant

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A.

Definition

For $n \geq 2$, the *determinant* of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Cofactor expansions

When $A = [a_{ij}]$, the (i, j)-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem 1

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any i and any j between 1 and n.

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The determinant of a triangular matrix

A *triangular* matrix is a square matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ when $i > j$.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

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Properties of determinants

Theorem 3

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
2. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

Properties of determinants

- Suppose an $n \times n$ matrix A has been reduced to an echelon form U by row replacements and row interchanges.
- If there are r interchanges, then $\det(A) = (-1)^r \det(U)$.
- Since U is in echelon form, it is triangular, so $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.
- If A is invertible, the entries $u_{11}, u_{22}, \ldots, u_{nn}$ are all pivots. Otherwise, at least one u_{ii} is zero.
- Thus,

$$\det(A) = \begin{cases} (-1)^r u_{11} u_{22} \ldots u_{nn} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

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Properties of determinants

Theorem 4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Column operations

Theorem 5

If A is a square matrix, then $\det(A^T) = \det(A)$.

Multiplicative property

Theorem 6

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

It follows from the theorem that if A and B are $n \times n$ matrices, then $\det(AB) = det(A)\det(B) = \det(B)\det(A) = \det(BA)$, even if $AB \neq BA$.

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Warnings

Let A and B be $n \times n$ matrices and let k be a scalar.

- In general, $\det(A + B) \neq \det(A) + \det(B)$.
- In general, $\det(kA) \neq k \det(A)$.

In fact, $\det(kA) = k^n \det(A)$.

Problem 4 from June 2005

Find the determinant of the matrix $A = \begin{bmatrix} -2 & 0 & 0 & 8 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

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Problem 6 from August 2010

For which values of the parameter a are the vectors $\mathbf{v}_1 = (1, -3, a)$, $\mathbf{v}_2 = (0, 1, a)$ and $\mathbf{v}_3 = (a, 2, 0)$ linearly dependent?

Problem 6 from June 2012

Let A be a 4×4 matrix. Let $B = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Assume that $\det(AB) = 4$. What is $\det(A)$?

Show that the equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has only the solution $x_1 = x_2 x_3 = x_4 = 0$.

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Tomorrow's lecture

Tomorrow we shall

- look at *Cramer's rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

Section 3.3 in "Linear Algebras and Its Applications" (pages 177–187).

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