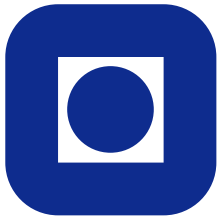

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TMA4115 - Calculus 3

Lecture 14, Feb 28

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Spring 2013

Yesterday's lecture

Yesterday we

- saw how we can *add* and *multiply* matrices,
- defined and studied *invertible* matrices.

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Today's lecture

Today we shall

- further study *invertible* matrices,
- look at *the invertible matrix theorem*.

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Matrix multiplication

Definition

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.
That is,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \ldots \ A\mathbf{b}_p].$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

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The inverse of a matrix

1. An $n \times n$ matrix A is said to be *invertible* if there exists an $n \times n$ matrix C such that $AC = CA = I_n$.
2. There can at most be one $n \times n$ matrix C such that $AC = CA = I_n$. If such a C exists, then it is called the *inverse* of A and is denoted by A^{-1}.
3. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix of T, then T is invertible if and only if A is.
4. If T and A are invertible, then A^{-1} is the standard matrix of T^{-1}.
5. A matrix that is not invertible is sometimes called a *singular* matrix, and an invertible matrix is called a *nonsingular* matrix.

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Using the inverse of a matrix to solve matrix equations

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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The inverse of an invertible 2×2 matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The number $ad - bc$ is called the *determinant* of A.

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Properties of invertible matrices

Theorem 6

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
2. If A and B are invertible $n \times n$ matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

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Elementary matrices

- An *elementary matrix* is one that is obtained by performing a single elementary row operation on an identity matrix.
- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.
- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

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Invertible matrices

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

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An algorithm for finding A^{-1}

1. Row reduce the augmented matrix $[A\ I]$.
2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.
3. If A is not row equivalent to I, then A is not invertible.

Example

Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$. Let us show that A is invertible and find A^{-1}.

The invertible matrix theorem

Theorem 8

Let A be a square $n \times n$ matrix. Then the following statements are equivalent, that is, for a given A, the following statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I_n.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

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7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
10. There is an $n \times n$ matrix C such that $CA = I_n$.
11. There is an $n \times n$ matrix D such that $AD = I_n$.
12. A^T is invertible.

Examples

Let us use the invertible matrix theorem to determine which of the following matrices are invertible.

$$\begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 8 \\ -3 & -2 & 5 \end{bmatrix}$$

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Problem 4 from the exam from December 2010

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Show that A is invertible and find A^{-1}.

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Problem 7 from the exam from August 2011

Let A and B be square $n \times n$ matrices and assume that AB is invertible. Show that A is also invertible.

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Example

Let E and F be $n \times n$ matrices and assume that $EF = I_n$. Let us show that E and F commute.

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Plan for next week

Wednesday we shall introduce and study *determinants*
Sections 3.1–3.2 in “Linear Algebras and Its Applications” (pages 163–177).

Thursday we shall

- look at *Cramer’s rule*,
- give a formula for the inverse of an invertible matrix,
- look at the relationship between areas, volumes and determinants.

Section 3.3 in “Linear Algebras and Its Applications” (pages 177–187).

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