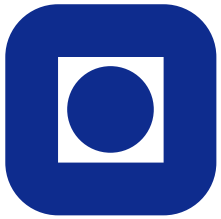

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TMA4115 - Calculus 3

Lecture 13, Feb 27

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Review of last week's lecture

Last week we looked at

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations,
- *applications* of linear models,
- *Maple* and *WolframAlpha*.

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Today's lecture

We shall

- see how we can *add* and *multiply* matrices,
- define and study *invertible* matrices.

Matrix addition and scalar multiplication

- If A and B are $m \times n$ matrices, then the *sum* $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries in A and B.
- The sum $A + B$ is defined only when A and B are the same size.
- If r is a scalar and A is an $m \times n$ matrix, then the *scalar multiple* rA is the $m \times n$ matrix whose entries are r times the corresponding entries in A.
- We will use $-A$ to denote the matrix $(-1)A$.

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Examples

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & -6 \\ 0 & 7 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$.

Then $A + B = \begin{bmatrix} 4 & 4 & -3 \\ -1 & 11 & 1 \end{bmatrix}$ and $-2C = \begin{bmatrix} -4 & -2 \\ 8 & -6 \end{bmatrix}$.

$A + C$ is not defined because A and C have different sizes.

Properties of matrix addition and scalar multiplication

Theorem 1

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ where 0 is the matrix of the same size of A whose entries are all zero.
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

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Matrix multiplication

Definition

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.
That is,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \ldots \ A\mathbf{b}_p].$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

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The standard matrix of the composition of 2 linear transformation

Let $T : \mathbb{R}^p \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. If A is the standard matrix of S, and B is the standard matrix of T, then AB is the standard matrix of $S \circ T$.

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Row—column rule for computing AB

If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B.
If A is an $m \times n$ matrix, B is an $n \times p$ matrix, a_{ik} denotes the (i,k)-entry in A, b_{kj} denotes the (k,j)-entry in B, and $(AB)_{ij}$ denotes the (i,j)-entry in A, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

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Properties of matrix multiplication

Theorem 2

Let r be a scalar, let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then:

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A I_n = A$

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AB and BA might be different

- BA might not be defined, even if AB is defined (if, for example A is a 2×2 matrix and B is a 2×3 matrix, then AB is defined, but BA is not).
- Even when AB and BA are both defined, then AB and BA might not have the same size (if, for example A is a 3×2 matrix and B is a 2×3 matrix, then AB is a 3×3 matrix and BA is a 2×2 matrix).
- Even when AB and BA are both defined, and AB and BA have the same size, then AB and BA might be different.

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Warnings

1. In general, $AB \neq BA$ (for instance, $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$).
2. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$ (for instance, $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$).
3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$ (for instance, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$).

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Commuting matrices

- If $AB = BA$, then we say that A and B *commute* (*kommutere* in Norwegian).
- A necessary (but not sufficient) condition for A and B to commute is that there are square matrices of the same size.

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Powers of a matrix

- If A is a square matrix and k is a positive integer, then A^k denotes the product of k copies of A. That is,

$$A^k = \underbrace{A \dots A}_{k \text{ times}}.$$

Example

In a certain region, 5% of a city's population moves to the surrounding suburbs each year, and 3% of the suburban population moves into the city.
We saw last week that if r_i denotes the population in the city in year $2000 + i$, s_i denotes the population in the suburbs in year $2000 + i$ and $\mathbf{x}_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}$, then $\mathbf{x}_{i+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \mathbf{x}_i$.

So $\mathbf{x}_n = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}^n \mathbf{x}_0$.

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The transpose of a matrix

Definition

Given an $m \times n$ matrix A, the *transpose* of A is the $n \times m$, denoted by A^T, whose columns are formed from the corresponding rows of A.

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Properties of the transpose

Theorem 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then:

1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$.
3. $(rA)^T = rA^T$ for any scalar r.
4. $(AB)^T = B^T A^T$.

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Invertible transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. T is said to be *invertible* if there exists a transformation $S : \mathbb{R}^m \to \mathbb{R}^n$ such that $S \circ T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ and $T \circ S(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y}$ in $\mathbb{R}^m$.
2. T is invertible if and only if it is onto and one-to-one.
3. There can at most be one transformation $S : \mathbb{R}^m \to \mathbb{R}^n$ such that $S \circ T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ and $T \circ S(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y}$ in $\mathbb{R}^m$. If such an S exists, then it is called the *inverse* of T and is denoted by T^{-1}.

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Invertible linear transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. A necessary (but not sufficient) condition for T to be onto is that $n \geq m$.
2. A necessary (but not sufficient) condition for T to be one-to-one is that $n \leq m$.
3. It follows that a necessary (but not sufficient) condition for T to be invertible is that $n = m$.
4. If T is invertible, then T^{-1} is a linear transformation.
5. If $n = m$ and A is the standard matrix of T, then T is invertible if and only if there exists a matrix C such that $AC = CA = I_n$.

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The inverse of a matrix

- An $n \times n$ matrix A is said to be *invertible* if there exists an $n \times n$ matrix C such that $AC = CA = I_n$.
- There can at most be one $n \times n$ matrix C such that $AC = CA = I_n$. If such a C exists, then it is called the *inverse* of A and is denoted by A^{-1}.
- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix of T, then T is invertible if and only if A is.
- If T and A are invertible, then A^{-1} is the standard matrix of T^{-1}.
- A matrix that is not invertible is sometimes called a *singular* matrix, and an invertible matrix is called a *nonsingular* matrix.

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Tomorrow's lecture

Tomorrow we shall

- further study *invertible* matrices,
- look at *the invertible matrix theorem*.

Section 2.3 in "Linear Algebras and Its Applications" (pages 111–116).

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