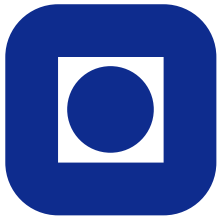

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TMA4115 - Calculus 3

Lecture 11, Feb 20

Toke Meier Carlsen
Norwegian University of Science and Technology

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Review of last week's lecture

Last week we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector form*,
- looked at applications of linear systems,
- introduced and studied *linear dependence* and *linear independence* of vectors.

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Today's lecture

We shall introduce and study

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations.

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Transformations

- A *transformation* (or *function* or *mapping*) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called domain of T, and $\mathbb{R}^m$ is called the codomain of T.
- The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the range (or image) of T.

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Transformations and real-valued functions of several variables

If $f_1, f_2, \ldots, f_m$ are real-valued functions on $\mathbb{R}^n$, then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

defines a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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Transformations and real-valued functions of several variables

Conversely, if T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there exist real-valued functions $f_1, f_2, \ldots, f_m$ on $\mathbb{R}^n$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

for all $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

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Matrix transformations

If A is an $m \times n$ matrix and we for every $\mathbf{x}$ in $\mathbb{R}^n$ let $T(x) = A\mathbf{x}$, then T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
The range of A is Span$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of A.

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Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in $\mathbb{R}^n$.

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Matrix transformations are linear

Every matrix transformation is linear because if A is an $m \times n$ matrix, then $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and $A(c\mathbf{u}) = cA\mathbf{u}$ for all scalars c and all $\mathbf{u}$ in $\mathbb{R}^n$.

Properties of linear transformations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then:

1. $T(\mathbf{0}) = \mathbf{0}$ because $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$ for any vector $\mathbf{u}$ in $\mathbb{R}^n$.
2. $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all scalars c and d and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, because $T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

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Example

A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on material, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product C, the company spends \$0.40 on material, \$0.30 on labor, and \$0.15 on overhead.
Let x_1 denote the value of product B and x_2 the value of product C that the company manufactores.
Let y_1, y_2 and y_3 denote the costs of material, labor and overhead the company spends on this.

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Example (cont.)

Then

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.45x_1 + 0.40x_2 \\ 0.25x_1 + 0.30x_2 \\ 0.15x_1 + 0.15x_2 \end{bmatrix}$$

So if we define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40 \\ 0.25 & 0.30 \\ 0.15 & 0.15 \end{bmatrix} \mathbf{x}$$

then $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = T(\mathbf{x})$.

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\-1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\5\\7\end{bmatrix}$.

1. Let $\mathbf{u} = \begin{bmatrix}2\\-1\end{bmatrix}$. Let us find the image $T(\mathbf{u})$ of $\mathbf{u}$ under T.
2. Let us find the image under T of $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$.
3. Is there more than one $\mathbf{x}$ whose image under T is $\begin{bmatrix}5\\1\\-9\end{bmatrix}$?

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Example (cont.)

4. Let $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Let us determine if $\mathbf{c}$ is in the range of T.

The matrix of a linear transformation

Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In fact, if we for each $j = 1, \ldots n$ let $\mathbf{e}_j$ be the jth column of the identity matrix I_n, then A is the $m \times n$ matrix $[T(\mathbf{e}_1) \ \ldots \ T(\mathbf{e}_n)]$ whose jth column is the vector $T(\mathbf{e}_j)$.

The matrix A is called the *standard matrix* of T.

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Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2, x_3) = (x_1 - 3x_2 - x_3, x_2 + 4x_3, 2x_1 + 9x_2 + 5x_3)$. Let us find the standard matrix of T.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $T(\mathbf{x}) = \mathbf{y}$ where $\mathbf{y}$ is the vector we obtain from rotating $\mathbf{x}$ by an angle of θ around zero.
Let us show that T is a linear transformation and find its standard matrix.

One-to-one transformations

Definition of one-to-one transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* (or *injective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that each $\mathbf{b}$ in $\mathbb{R}^m$ does not have to be in the image of T in order for T to be one-to-one.

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One-to-one transformations

Theorem 11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

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Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if T is one-to-one.

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How to determine if a linear transformation is one-to-one

The following procedure outlines how to determine if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one:

1. Find the standard matrix A of T.
2. Reduce A to an echelon form.
3. If A has a pivot position in every column, then the equation $A\mathbf{x} = \mathbf{0}$ has no free variable, so the equation $A\mathbf{x} = \mathbf{0}$ only has the trivial solution, and T is one-to-one.

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Onto transformations

Definition of onto transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* (or *surjective*) if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Notice that a transformation T is onto if and only if the image of T is all of $\mathbb{R}^m$.

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Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 2 & 6 & -3 \\ -1 & 2 & 4 & 5 \end{bmatrix}$$

Let us determine if T is onto.

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How to determine if a linear transformation is onto

The following procedure outlines how to determine if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto:

1. Find the standard matrix A of T.
2. Reduce A to an echelon form.
3. If A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{x}$ in $\mathbb{R}^m$ and T is onto.

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Onto and one-to-one linear transformations

Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
2. T is one-to-one if and only if the columns of A are linearly independent.

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 + 2x_2, 2x_1 + x_2, 0)$. Let us determine if T is onto and if it is one-to-one.

Tomorrow's lecture

Tomorrow we shall

- look at *applications* of linear models,
- look at the use of *Maple* and *WolframAlpha*.

Section 1.10 in "Linear Algebras and Its Applications" (pages 80-90).

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