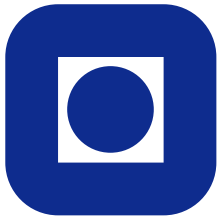

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TMA4115 - Calculus 3

Lecture 10, Feb 14

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday's lecture

Yesterday we

- introduced and solved *homogeneous* and *nonhomegeneous* matrix equations,
- learned how to write solution sets in *parametric vector form*,
- looked at applications of linear systems.

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Today's lecture

We shall introduce and study *linear dependence* and *linear independence* of vectors.

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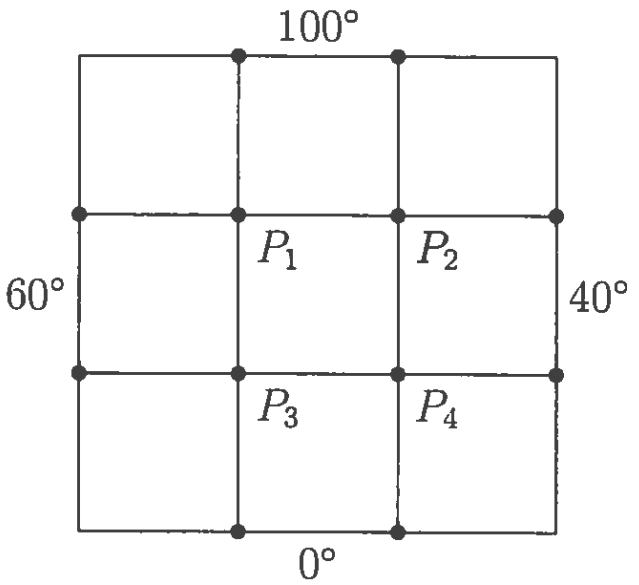
Problem 4 from December 2007

Problem 4 We shall determine the temperatures T_1, T_2, T_3, T_4 at four points P_1, P_2, P_3, P_4 on the square plate, see the figure on the right. Temperatures along the sides of the plate are shown on the figure.

Set up and solve a system of equations for T_1, T_2, T_3, T_4 if we assume that the temperature at each point P_1 to P_4 is the average of the temperatures at the four neighboring points (to the right, above, to the left, and below).



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Linear independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

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Linear dependence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly dependent* if there exist weights $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

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Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Let us determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

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Linear independence of matrix columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Thus we have the following important fact.

Linear independence of matrix columns

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial equation.

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Example

Let us determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$ are linearly independent.

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Linear dependence of sets containing the zero vector

Theorem 9

If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

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Sets of one vector

A set containing only one vector **v**, is linearly independent if and only if **v** is not the zero vector.

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Sets of two vectors

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.

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Sets of two or more vectors

Theorem 7: Characterization of linearly dependent sets

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
In fact, if S is linearly dependent and $\mathbf{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Sets of two or more vectors

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, and let $\mathbf{w}$ be an unknown vector in $\mathbb{R}^3$.

- If $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent because $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent because
  1. $\mathbf{u}$ is not the zero vector,
  2. $\mathbf{v}$ is not a linear combination of $\mathbf{u}$ ($\mathbf{v}$ is not a multiple of $\mathbf{u}$),
  3. $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$ because $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$.

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Sets of two or more vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

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Example

The following sets are all linearly dependent:

1. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$;
2. $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$;
3. $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

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Problem 6 from August 2010

For which values of the parameter a are the vectors $\mathbf{v}_1 = (1, -3, a)$, $\mathbf{v}_2 = (0, 1, a)$ and $\mathbf{v}_3 = (a, 2, 0)$ linearly dependent?

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Problem 5 from December 2007

Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent vectors in $\mathbb{R}^n$. Are vectors

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_3,\ \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3$$

linearly dependent or linearly independent? (Remember to explain the answers.)

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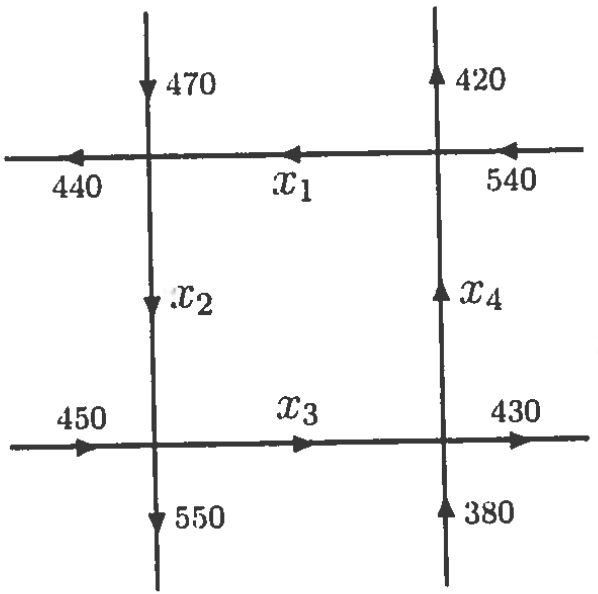
Problem 4 from December 2006

Problem 4 Four one-way streets in a city intesect as shown in the figure to the right. The number of cars passing per hour is shown in the figure.

Write down a system of equations in the form $A\mathbf{x} = \mathbf{b}$ which $\mathbf{x} = (x_1, x_2, x_3, x_4)$ must satisfy, and solve the system.

Find x_1, x_2 and x_3 if the x_4-section is closed for traffic, making $x_4 = 0$.



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Problem 6 from August 2005

Let A be an $m \times n$ matrix where $m > n$. Explain why there is a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

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Plan for next week

Wednesday we shall introduce and study

- *linear transformations*,
- the *standard matrix* of a linear transformation,
- *onto* linear transformations,
- *one-to-one* linear transformations,

Section 1.8-1.9 in “Linear Algebras and Its Applications” (pages 62-80).

Thursday we shall look at *applications of linear models*. Section 1.10 in “Linear Algebras and Its Applications” (pages 80-90).

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