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TMA4115 - Calculus 3

Lecture 1, Jan 16

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Course web page

Information about the course can be found at
`http://wiki.math.ntnu.no/tma4115/2013v`.

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Overview of the course

The topics of the course are

- Complex numbers (1.5 weeks)
- Second order linear differential equations (1.5 weeks)
- Linear algebra (10 weeks)

Complex numbers

- A complex number is a number which can be written as $a + ib$ where a and b are real numbers and i satisfies $i^2 = -1$.
- The Italian mathematician Gerolamo Cardano is credited with introducing complex numbers in around 1545 in order to solve cubic equations.
- Complex numbers are not only used for solving polynomial equations, but naturally show up all over in mathematics and are used in many scientific fields, including engineering, electromagnetism and quantum physics.

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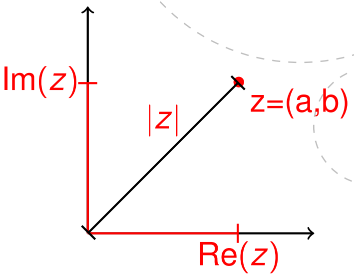
Complex numbers as points in the plane

- A complex number can be represented as a point (a, b) in the plane.
- If $z = (a, b)$, then a is called the *real part* of z and is denoted by $\mathrm{Re}(z)$.
- b is called the *imaginary part* of z and is denoted by $\mathrm{Im}(z)$.
- The length $\sqrt{a^2 + b^2}$ of the line from $(0, 0)$ to (a, b) is called the *modulus* or the *absolute value* of z and is denoted by $|z|$.



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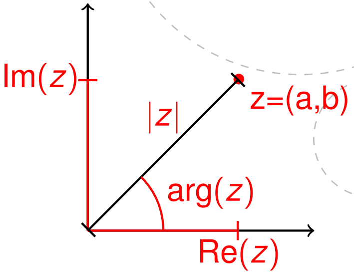
Complex numbers as points in the plane

- The angle between the line through $(0,0)$ and (a,b) and the positive part of the real axis is called the *argument* of z and is denoted by $\arg(z)$.
- $\arg(z)$ is not unique. If $\theta = \arg(z)$, then also $\theta + 2\pi = \arg(z)$.



If we want to be precise, then $\arg(z)$ is really the set of all angles θ which satisfies that if we rotate the positive part of the real axis by θ, then it lands on the line through $(0,0)$ and (a,b).

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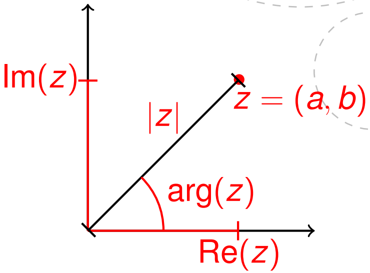
Complex numbers as points in the plane

- The unique value of $\arg(z)$ in the interval $(-\pi, \pi]$ is called the *principal argument* of z and is denoted by $\text{Arg}(z)$.
- Notice that $\arg(z)$ and $\text{Arg}(z)$ are not defined if $z = (0, 0)$.



How to compute the argument of a complex number

The argument of a complex number z can be computed using trigonometric identities, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$ and arccos, arcsin and arctan.

- If $\mathrm{Re}(z) > 0$, then
  $\mathrm{Arg}(z) = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) = \arcsin\left(\frac{\mathrm{Im}(z)}{|z|}\right)$.
- If $\mathrm{Im}(z) \geq 0$, then $\mathrm{Arg}(z) = \arccos\left(\frac{\mathrm{Re}(z)}{|z|}\right)$.
- $\mathrm{Arg}(z) = \mathrm{Arg}(-z) \pm \pi$.

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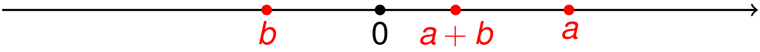
How do we add real numbers?

If a and b are real numbers, then we get $a+b$ by taking the line from 0 to b and move it such that it starts at a.



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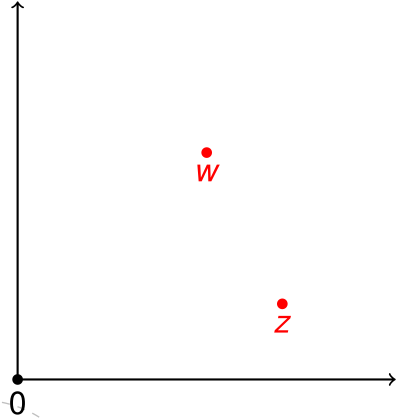
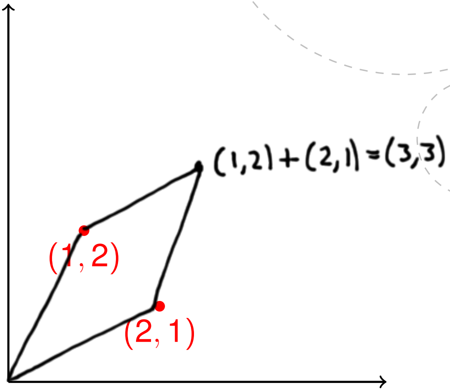
How do we add complex numbers?

If z and w are complex numbers, then we get $z + w$ by taking the line from 0 to w and move it such that it starts at z.



Example

What is $(2,1) + (1,2)$?



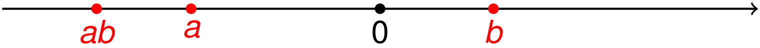
Addition of complex numbers

If z_1, z_2 and z_3 are complex numbers, then

- $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$,
- $z_1 + z_2 = z_2 + z_1$,
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$,
- $|z_1 + z_2| \leq |z_1| + |z_2|$.

How do we multiply real numbers?

If a and b are real numbers and $b > 0$, then we get ab by scaling the line from 0 to a by a factor of b.



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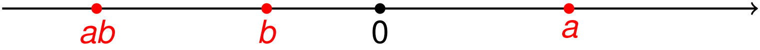
How do we multiply real numbers?

If a and b are real numbers and $b < 0$, then we get ab by rotating the line from 0 to a by 180° degrees and then scale it by a factor of $|b|$.



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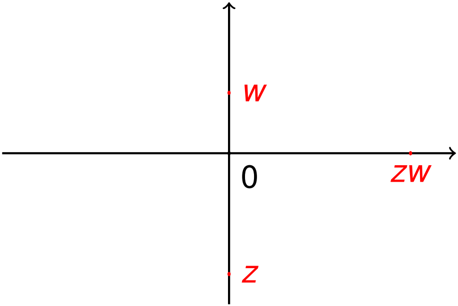
How do we multiply complex numbers?

If z and w are real numbers, then we get zw by rotating the line from 0 to z by $\arg(w)$ and then scale it by a factor of $|w|$.



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Multiplication of complex numbers

If z_1, z_2 and z_3 are complex numbers, then

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $|z_1 z_2| = |z_1||z_2|$,
- $z_1 z_2 = z_2 z_1$,
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$,
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Powers of complex numbers

If z is a complex number and n is a positive integer, then $\arg(z^n) = n \arg(z)$ and $|z^n| = |z|^n$.

Notation

- When a is a real number, then we identify $(a, 0)$ with a.
- Let $i = (0, 1)$.
- Then every complex number (a, b) can be written as $a + bi$.
- $\text{Re}(a + bi) = a$, $\text{Im}(a + bi) = b$ and $|a + bi| = \sqrt{a^2 + b^2}$.
- $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- $i^2 = -1$
- $(a + bi)(c + di) = ac - bd + (ad + bc)i$.

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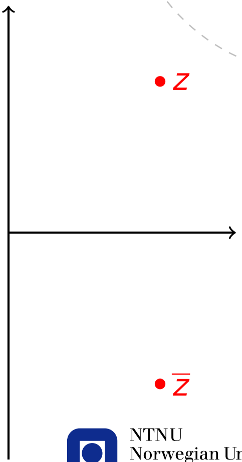
Complex conjugation

- When $z = a + bi$ is a complex number, then the number $a - bi$ is called the *conjugate* of z and is denoted by $\overline{z}$.
- $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$.
- We get $\overline{z}$ by reflecting z in the real line.
- $|\overline{z}| = |z|$ and $\arg(\overline{z}) = -\arg(z)$.



Complex conjugation

- $z = \bar{z}$ if and only if $\text{Im}(z) = 0$.
- $z = -\bar{z}$ if and only if $\text{Re}(z) = 0$.
- $\overline{z + w} = \bar{z} + \bar{w}$.
- $\overline{zw} = \bar{z}\,\bar{w}$.
- $z\bar{z} = |z|^2$.
- $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$.

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Problem 1 from the exam from June 2009

Find all complex numbers $z = x + iy$ which satisfy the equality $|z + 1 - i\sqrt{3}| = |z - 1 + i\sqrt{3}|$. Draw the solutions in a diagram.

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Problem 1 from the exam from August 2011

Find all complex numbers z such that $\mathrm{Im}(-z + i) = (z + i)^2$. Draw the solutions on a diagram.