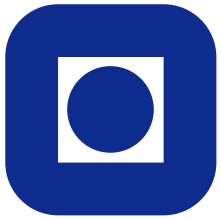

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TMA4115 - Calculus 3

Lecture 4, Jan 24

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Review of yesterday's lecture

Yesterday we

- looked at how to use complex numbers to solve polynomial equations,
- looked at *the fundamental theorem of algebra*,
- introduced *the complex exponential function*,
- and studied extensions of trigonometric functions to the complex numbers.

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Today's lecture

Today we shall

- study second-order linear differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.

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Second-order differential equations

A second-order differential equation is a differential equation which can be written on the form

$$y'' = f(t, y, y').$$

A solution to such an equation is a twice continuously differentiable function $y(t)$ satisfying

$$y''(t) = f(t, y(t), y'(t)).$$

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Examples of second-order differential equations

- $y'' + \cos(y) = e^t$.
- $y'' + 5y = 0$.
- $t^2 y'' + \sin(t) y = 3$.

Second-order linear differential equations

A second-order linear differential equation is a differential equation with can be written on the form

$$y'' + p(t)y' + q(t)y = g(t).$$

Examples of second-order differential equations

- $y'' + \cos(y) = e^t$ is not linear.
- $y'' + 5y = 0$ is linear.
- $t^3 y'' + \sin(t) y = 3$ is linear.

Existence and uniqueness of solutions to a second-order linear equation

Suppose the functions p, q and g are continuous on the interval (α, β). Let t_0 be any point in (α, β). Then for any real numbers a and b there is one and only one function defined on (α, β) which is a solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

on (α, β) and satisfies the initial conditions $y(t_0) = a$ and $y'(t_0) = b$.

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Second-order linear homogeneous differential equations

A second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is *homogeneous* if $g = 0$. If $g \neq 0$, then the differential equation is called *inhomogeneous* or *nonhomegeneous*.

Examples

- $y'' + 5y = 0$ is homogeneous.
- $t^3y'' + \sin(t)y = 3$ is inhomogeneous.

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The superposition principle

If y_1 and y_2 are solutions to the second-order homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then so is $y(t) = c_1 y_1(t) + c_2 y_2(t)$ for any choice of constants c_1 and c_2.

Linear combinations

- When y_1 and y_2 are functions and c_1 and c_2 are constants, then the function $c_1 y_1(t) + c_2 y_2(t)$ is a *linear combination* of y_1 and y_2.
- So the previous results says that if y_1 and y_2 are solutions to a second-order homogeneous linear differential equation, then any linear combination of y_1 and y_2 is also a solution to the same differential equation.

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Example

Let us find the solution of the differential equation

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0 \tag{1}$$

on the interval $(0, \infty)$ which satisfies that $y(2) = 8$ and $y'(2) = 0$.
Let $y(t) = t^n$. Then

$$\begin{aligned} y''(t) - \frac{4}{t}y'(t) + \frac{6}{t^2}y(t) &= n(n-1)t^{n-2} - 4nt^{n-2} + 6t^{n-2} \\ &= (n^2 - 5n + 6)t^{n-2} \\ &= (n-2)(n-3)t^{n-2}. \end{aligned}$$

So y is a solution of (1) if and only if $n = 2$ or $n = 3$.

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Example

Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then it follows from the superposition principle and the calculations above that y is a solution of (1).

$$\begin{aligned} y'(t) &= 2c_1 t + 3c_2 t^2, \\ y(2) &= 4c_1 + 8c_2, \\ y'(2) &= 4c_1 + 12c_2, \end{aligned}$$

so $y(2) = 8$ and $y'(2) = 0$ if and only if $4c_1 + 8c_2 = 8$ and $4c_1 + 12c_2 = 0$.

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Example

The solution of the linear system

$$4c_1 + 8c_2 = 8$$
$$4c_1 + 12c_2 = 0$$

is $c_1 = 6$ and $c_2 = -2$, so the solution of (1) which satisfies that $y(2) = 8$ and $y'(2) = 0$ is $y(t) = 6t^2 - 2t^3$.

Example

Let $t_0 \in (0, \infty)$ and $a, b \in \mathbb{R}$. Let us now find the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$.
Let $y(t) = c_1 t^2 + c_2 t^3$ where c_1 and c_2 are constants. Then

$$y(t_0) = c_1 t_0^2 + c_2 t_0^3,$$
$$y'(t_0) = 2c_1 t_0 + 3c_2 t_0^2,$$

so $y(t_0) = a$ and $y'(t_0) = b$. if and only if $c_1 t_0^2 + c_2 t_0^3 = a$ and $2c_1 t_0 + 3c_2 t_0^2 = b$.

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Example

The solution of the linear system

$$c_1 t_0^2 + c_2 t_0^3 = a$$
$$2c_1 t_0 + 3c_2 t_0^2 = b$$

is $c_1 = \frac{3at_0^2 - bt_0^3}{t_0^4}$ and $c_2 = \frac{bt_0^2 - 2at_0}{t_0^4}$, so the solution of (1) which satisfies that $y(t_0) = a$ and $y'(t_0) = b$ is
$y(t) = \frac{3at_0^2 - bt_0^3}{t_0^4} t^2 - \frac{bt_0^2 - 2at_0}{t_0^4} t^3$.
Notice that $t_0^4 = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$ where $y_1(t) = t^2$ and $y_2(t) = t^3$.

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The Wronskian

Let u and v be two differential functions. The *Wronskian* of u and v is the function

$$W(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t).$$

Solutions to initial value problems

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

on the interval (α, β). Let t_0 be a point in the interval (α, β) and let a and b be arbitrary real numbers. If $W(t)$ is the Wronskian of y_1 and y_2 and $W(t_0) \neq 0$, then there exist constants c_1 and c_2 such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the unique solution to (2) on (α, β) which satisfies $y(t_0) = a$ and $y'(t_0) = b$.

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Proof

It follows from the superposition principle that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) for any choice of constants c_1 and c_2.
$y(t_0) = a$ and $y'(t_0) = b$ if and only if $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.
Since $y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = W(t_0) \neq 0$, the system

$$c_1 y_1(t_0) + c_2 y_2(t_0) = a$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$$

has a solution, so there exist constants c_1 and c_2 such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2) on (α, β) which satisfies $y(t_0) = a$ and $y'(t_0) = b$.

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Proposition 1.26

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then the Wronskian of y_1 and y_2 is either identically equal to zero on the interval (α, β), or it is never equal to zero on the interval (α, β).

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Proof

If we differentiate the Wronskian W we get

$$\begin{aligned} W'(t) &= (y_1(t)y_2'(t) - y_1'(t)y_2(t))' \\ &= y_1'(t)y_2'(t) + y_1(t)y_2''(t) - y_1''(t)y_2(t) - y_1'(t)y_2'(t) \\ &= y_1(t)y_2''(t) - y_1''(t)y_2(t). \end{aligned}$$

Since y_1 and y_2 are solutions to the differential equation $y'' + py' + qy = 0$, we have that $y_1''(t) = -p(t)y_1'(t) - q(t)y_1$ and $y_2''(t) = -p(t)y_2'(t) - q(t)y_2$.

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Proof

It follows that

$$\begin{aligned} W'(t) &= y_1(t)y_2''(t) - y_1''(t)y_2(t) \\ &= -p(t)y_1(t)y_2'(t) - q(t)y_1(t)y_2(t) \\ &\quad + p(t)y_1'(t)y_2(t) + q(t)y_1(t)y_2(t) \\ &= -p(t)y_1(t)y_2'(t) + p(t)y_1'(t)y_2(t) \\ &= -p(t)(_1(t)y_2'(t) - y_1'(t)y_2(t)) \\ &= -p(t)W(t), \end{aligned}$$

so W is a solution to the first-order differential equation

$$W' = -pW.$$

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Proof

If t_0 is a point in (α, β), the solution to this equation is

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}.$$

It follows that if $W(t_0) = 0$, then $W(t) = 0$ for all t in (α, β), and if $W(t_0) \neq 0$, then $W(t) \neq 0$ for all t in (α, β).

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Linear dependent functions

- Two functions u and v are *linear dependent* on an interval (α, β) if there exist two constants c_1 and c_2, which are no both zero, such that $c_1 u(t) + c_2 v(t) = 0$ for all $t \in (\alpha, \beta)$.
- u and v are *linear independent* if they are not linear dependent.
- If $u = cv$ for some constant c, then u and v are linear dependent because $u - cv = 0$.
- Conversely, if u and v are linear dependent, then $c_1 u + c_2 v = 0$ for some choice of constants c_1 and c_2 which are no both zero, and then $u = -(c_2/c_1)v$ if $c_1 \neq 0$, and $v = -(c_1/c_2)u$ if $c_2 \neq 0$.

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Linear dependent functions

Two functions u and v are linear dependent if and only if u is a constant multiple of v, or v is a constant multiple of u.

Proposition 1.27

Suppose the functions y_1 and y_2 are solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β).
Then y_1 and y_2 are linearly dependent on (α, β) if and only if the Wronskian of y_1 and y_2 is identically equal to zero on the interval (α, β).

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Proof

If $y_1(t) = cy_2(t)$ for some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_2(t)y_2'(t) - cy_2'(t)y_2(t) = 0.$
Similarly, if $y_2(t) = cy_1(t)$ some constant c, then
$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = cy_1(t)y_1'(t) - cy_1'(t)y_1(t) = 0.$
Conversely, suppose that $W(t) = 0$ for all t in (α, β). If $y_2(t) = 0$, then $y_2(t) = 0y_1(t)$. If $y_2(t_0) \neq 0$ for some t_0 in (α, β), then it follows from the continuity of y_2 that there is a subinterval (c, d) of (α, β) which contains t_0 such that $y_2(t) \neq 0$ for all t in (c, d). On this interval we have

$$\frac{d}{dt}(y_1(t)/y_2(t)) = \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{(y_2(t))^2} = \frac{-W(t)}{(y_2(t))^2} = 0.$$

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When is the Wronskian zero?

Hence, on the interval (c, d), $y_1(t)/y_2(t)$ is equal to a constant c, and $y_1(t) = cy_2(t)$. In particular, $y_1(t_0) = cy_2(t_0)$ and $y_1'(t_0) = cy_2'(t_0)$.
Since both $y_1(t)$ and $cy_2(t)$ are solutions to the initial value problem

$$y'' + p(t)y' + q(t)y = 0,\ y(t_0) = y_1(t_0),\ y'(t_0) = y_1'(t_0)$$

on (α, β), it follows that $y_1(t) = cy_2(t)$ on (α, β).

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Theorem 1.23

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

on the interval (α, β). Then any solution to the equation is a linear combination of y_1 and y_2.

Proof

Since y_1 and y_2 are linearly independent, neither is a constant multiple of the other, so the Wronskian W of y_1 and y_2 is not zero at any point.
Let y be a solution to the equation. Pick a point t_0 in (α, β) and let $a = y(t_0)$ and $b = y'(t_0)$.
Since $W(t_0) \neq 0$ there exist constants c_1 and c_2 such that $c_1 y_1(t_0) + c_2 y_2(t_0) = a$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = b$.
Then both y and $c_1 y_1 + c_2 y_2$ are solutions to the initial value problem $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = a$, $y'(t_0) = b$ on the interval (α, β). It follows that $y = c_1 y_1 + c_2 y_2$.

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Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,
  2. or by showing that the Wronskian of y_1 and y_2 is not zero at any point.

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Second-order homogeneous linear differential equations with constant coefficients

A second-order homogeneous linear differential equation with constant coefficients is a differential equation with can be written on the form

$$y'' + py' + qy = 0$$

where p and q are constants.

Finding solutions to second-order homogeneous linear differential equations with constant coefficients

Inserting $y(t) = e^{\lambda t}$ into the differential equation, we obtain

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t}.$$

Since $e^{\lambda t} \neq 0$, we have that $y(t) = e^{\lambda t}$ is a solution if and only if $\lambda^2 + p\lambda + q = 0$.

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Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

1. $p^2 - 4q > 0$,
2. $p^2 - 4q < 0$,
3. $p^2 - 4q = 0$.

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Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) = \lambda_2 e^{\lambda_1 t}e^{\lambda_2 t} - \lambda_1 e^{\lambda_2 t}e^{\lambda_1 t} \\ &= (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants.

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Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 3\lambda + 2$, and the characteristic roots are $\lambda = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$.
It follows that the general solution is $y(t) = c_1 e^{2t} + c_2 e^{t}$.

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2 which have the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ where a and b are real numbers and $b \neq 0$.
The 2 functions $z_1(t) = e^{\lambda_1 t} = e^{at}(\cos(bt) + i\sin(bt))$ and $z_2(t) = e^{\lambda_2 t} = e^{at}(\cos(bt) - i\sin(bt))$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The two functions $y_1(t) = \mathrm{Re}(z_1(t)) = e^{at}\cos(bt)$ and $y_2(t) = \mathrm{Im}(z_1(t)) = e^{at}\sin(bt)$ are also solutions to the differential equation $y'' + py' + qy = 0$.

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Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$\begin{aligned} W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ =& e^{at}\cos(bt)(ae^{at}\sin(bt) + be^{at}\cos(bt)) \\ & - e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt)) \\ =& be^{2at}\cos^2(bt) + be^{2at}\sin^2(bt) \\ =& be^{2at} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form
$c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$ where c_1 and c_2 are constants.

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Example

Let us find the general solution to the second-order differential equation $y'' + 2y' + 2y = 0$.
The characteristic polynomial is $\lambda^2 + 2\lambda + 2$, and the characteristic roots are
$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm -4}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$.
It follows that the general solution is
$y(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$.

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$\begin{aligned} y_2''(t) + py_2'(t) + qy_2(t) &= 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + p(e^{\lambda t} + \lambda te^{\lambda t}) + qte^{\lambda t} \\ &= (2\lambda + p)e^{\lambda t} + (\lambda^2 + p\lambda + q)te^{\lambda t} \\ &= 0. \end{aligned}$$

So y_2 is a solution to the differential equation $y'' + py' + qy = 0$.

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Repeated roots

The Wronskian of $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ is

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ &= e^{\lambda t}(e^{\lambda t} + \lambda te^{\lambda t}) - \lambda e^{\lambda t} te^{\lambda t} \\ &= e^{2\lambda t} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 e^{\lambda t} + c_2 te^{\lambda t}$ where c_1 and c_2 are constants.

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Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.
The characteristic polynomial is $\lambda^2 - 2\lambda + 1$, and the characteristic roots are $\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$.
It follows that the general solution is $y(t) = c_1 e^t + c_2 t e^t$.

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Exercise 4.3.26

Find the solution to the initial value problem

$$10y'' - y' - 3y = 0,\ y(0) = 1,\ y'(0) = 0.$$

Solution

The characteristic polynomial is $10\lambda^2 - \lambda - 3$, and the characteristic roots are $\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} \begin{cases} 12/20 \\ -10/20 \end{cases}$.

It follows that the general solution to the equation $10y'' - y' - 3y = 0$ is $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$.

Let $y(t) = c_1 e^{3t/5} + c_2 e^{-t/2}$. Then $y'(t) = (3/5)c_1 e^{3t/5} - (c_2/2)e^{-t/2}$, $y'(0) = 3c_1/5 - c_2/2$, and $y(0) = c_1 + c_2$.

So y is a solution to the initial value problem if and only if $c_1 + c_2 = 1$ and $3c_1/5 - c_2/2 = 0$.

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Solution

The solution to the linear system

$$c_1 + c_2 = 1$$
$$3c_1/5 - c_2/2 = 0$$

is $c_1 = 5/11$ and $c_2 = 6/11$, so the solution to the initial value problem

$$10y'' - y' - 3y = 0,\ y(0) = 1,\ y'(0) = 0.$$

is $y(t) = (5/11)e^{3t/5} + (6/11)e^{-t/2}$.

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Plan for next week

Wednesday we shall

- study *harmonic motions*,
- study solutions of *second-order linear inhomogeneous differential equations*,
- look at the *method of undetermined coefficients*.

Section 4.4 and 4.5 (pages lv–lxxii).
Thursday we shall

- look at *variation of parameters*,
- study *forced harmonic motions*.

Section 4.6 and 4.7 (pages lxxii–lxxxvi).

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