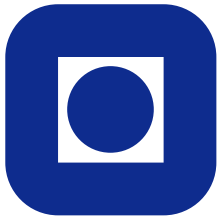


NTNU
Norwegian University of
Science and Technology

# TMA4115 - Calculus 3

## Lecture 3, Jan 23

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

# Review of last week

Last week we

- introduced complex numbers, both in a geometric way and in an algebraic way,
- defined $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$ and $\arg(z)$ for a complex number $z$,
- defined addition and multiplication of complex numbers,
- defined complex conjugation,
- introduced polar representation of complex numbers,
- computed powers of complex numbers,
- defined and computed roots of complex numbers.

NTNU
Norwegian University of
Science and Technology

# Today's lecture

Today we shall

- use complex numbers to solve polynomial equations,
- look at *the fundamental theorem of algebra*,
- introduce *the complex exponential function*,
- and study extensions of trigonometric functions to the complex numbers.

NTNU
Norwegian University of
Science and Technology

# Solutions to second degree equations

If $a, b, c$ are complex numbers and $a \neq 0$, then the solutions to the equation $az^2 + bz + c = 0$ are $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

# Example

Let us solve the equation $z^2 + 4iz - 4 - 4i = 0$.

# Solution

The solutions are $z = \dfrac{-4i \pm \sqrt{(4i)^2 - 4(-4-4i)}}{2}$, so we must compute the square roots of
$(4i)^2 - 4(-4-4i) = -16 + 16 + 16i = 16i$.
Since $|16i| = 16$ and $\mathrm{Arg}(16i) = \frac{\pi}{2}$, the two square roots of $16i$ are

$$w_0 = 4\big(\cos(\pi/4) + i\sin(\pi/4)\big) = 4\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$= 2\sqrt{2} + i2\sqrt{2}$$

$$w_1 = 4\big(\cos(5\pi/4) + i\sin(5\pi/4)\big) = 4\left(\frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}\right)$$
$$= -2\sqrt{2} - i2\sqrt{2}$$

NTNU
Norwegian University of
Science and Technology

# Solution

So the two solutions of $z^2 + 4iz - 4 - 4i = 0$ are

$$z = \frac{-4i \pm \sqrt{(4i)^2 - 4(-4-4i)}}{2} = \frac{-4i \pm \sqrt{16i}}{2}$$

$$= \frac{-4i \pm (2\sqrt{2} + i2\sqrt{2})}{2} = \begin{cases} \sqrt{2} + (\sqrt{2} - 2)i \\ -\sqrt{2} - (\sqrt{2} + 2)i. \end{cases}$$

NTNU
Norwegian University of
Science and Technology

# Problem 1 from the exam from June 2012

Solve $w^2 = (-1 + i\sqrt{3})/2$. Find all solutions of the equation $z^4 + z^2 + 1 = 0$ and draw them in the complex plane. Write the solutions in the form $x + iy$.

NTNU
Norwegian University of
Science and Technology

# Solution

$|(-1+i\sqrt{3})/2| = \sqrt{(-1/2)^2+(\sqrt{3}/2)^2} = \sqrt{1/4+3/4} = 1$ and $\mathrm{Arg}((-1+i\sqrt{3})/2) = \arccos(-1/2) = \frac{2\pi}{3}$, so the solutions of $w^2 = (-1+i\sqrt{3})/2$ are

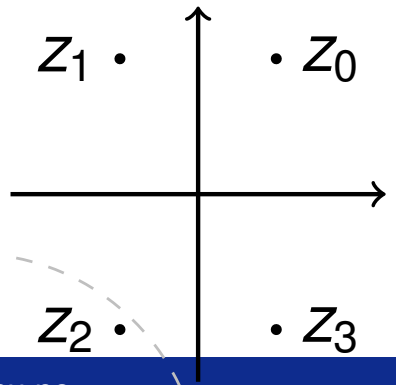
$$w_0 = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$w_1 = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

NTNU
Norwegian University of
Science and Technology

# Solution

$z^4 + z^2 + 1 = 0$ if and only if $z^2 = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$. It follows from the first part that $z^2 = (-1 + i\sqrt{3})/2$ if and only if $z = \pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. We have that $z^2 = (-1 - i\sqrt{3})/2$ if and only if $(\overline{z})^2 = (-1 + i\sqrt{3})/2$, and it follows from the first part that $(\overline{z})^2 = (-1 + i\sqrt{3})/2$ if and only if $\overline{z} = \pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. So the four solutions of the equation $z^4 + z^2 + 1 = 0$ are $z = \pm\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$.




NTNU
Norwegian University of
Science and Technology

# Roots of polynomials

If $P(z)$ is a polynomial, then a solution to the equation $P(z) = 0$ is called a *root* (or *zero*) of $P(z)$.
$z_0$ is a root of $P(z)$ if and only if $(z - z_0)$ is a factor of $P(z)$ (i.e., if $P(z) = (z - z_0)Q(z)$ for some polynomial $Q(z)$).

# The fundamental theorem of algebra

Every complex polynomial of degree 1 or higher has a least one complex root.

NTNU
Norwegian University of
Science and Technology

# Roots of real polynomials

If $w$ is a root of a real polynomial $\sum_{k=0}^{n} a_k z^k$, then $\overline{w}$ is also a root of $\sum_{k=0}^{n} a_k z^k$.

# Exercise 34 on page xxvii

Check that $z_1 = 1 - \sqrt{3}i$ is a zero of $P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16$, and find all the zeros of $P$.

# Solution

$z_1^2 = (1-\sqrt{3}i)(1-\sqrt{3}i) = 1-\sqrt{3}i-\sqrt{3}i-3 = -2-2\sqrt{3}i,$
$z_1^3 = (-2-2\sqrt{3}i)(1-\sqrt{3}i) = -2+2\sqrt{3}i-2\sqrt{3}i-6 = -8,$
and $z_1^4 = -8(1-\sqrt{3}i) = -8+8\sqrt{3}i$, so

$$z_1^4 - 4z_1^3 + 12z_1^2 - 16z_1 + 16$$
$$= -8+8\sqrt{3}i+32-24-24\sqrt{3}i-16+16\sqrt{3}i+16 = 0.$$

Since $z_1$ is a zero of $P$ and $P$ has real coefficients, it follows that also $\overline{z_1} = 1+\sqrt{3}i$ is a zero of $P$.
It follows that $(z-z_1)(z-\overline{z_1}) = z^2-(z_1+\overline{z_1})+z_1\overline{z_1} = z^2-2\,\mathrm{Re}(z_1)z+|z_1|^2 = z^2-2z+4$ is a factor of $P(z)$.

NTNU
Norwegian University of
Science and Technology

# Solution

By doing polynomial division, we see that
$P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16 = (z^2 - 2z + 4)(z^2 - 2z + 4)$,
so $z_1$ and $\overline{z_1}$ are the only zeros of $P$.

NTNU
Norwegian University of
Science and Technology

# Complex functions

A complex function $f$ is a rule that assigns a unique complex number $f(z)$ to each number $z$ in some set of complex numbers (called the *domain* of $f$).

## Examples of complex functions

- $f(z) = \mathrm{Re}(z)$
- $g(z) = \mathrm{Im}(z)$
- $h(z) = |z|$
- $j(z) = \mathrm{Arg}(z)$
- $k(z) = \overline{z}$
- $p(z) = z^2 - 4z + 6$

NTNU
Norwegian University of
Science and Technology

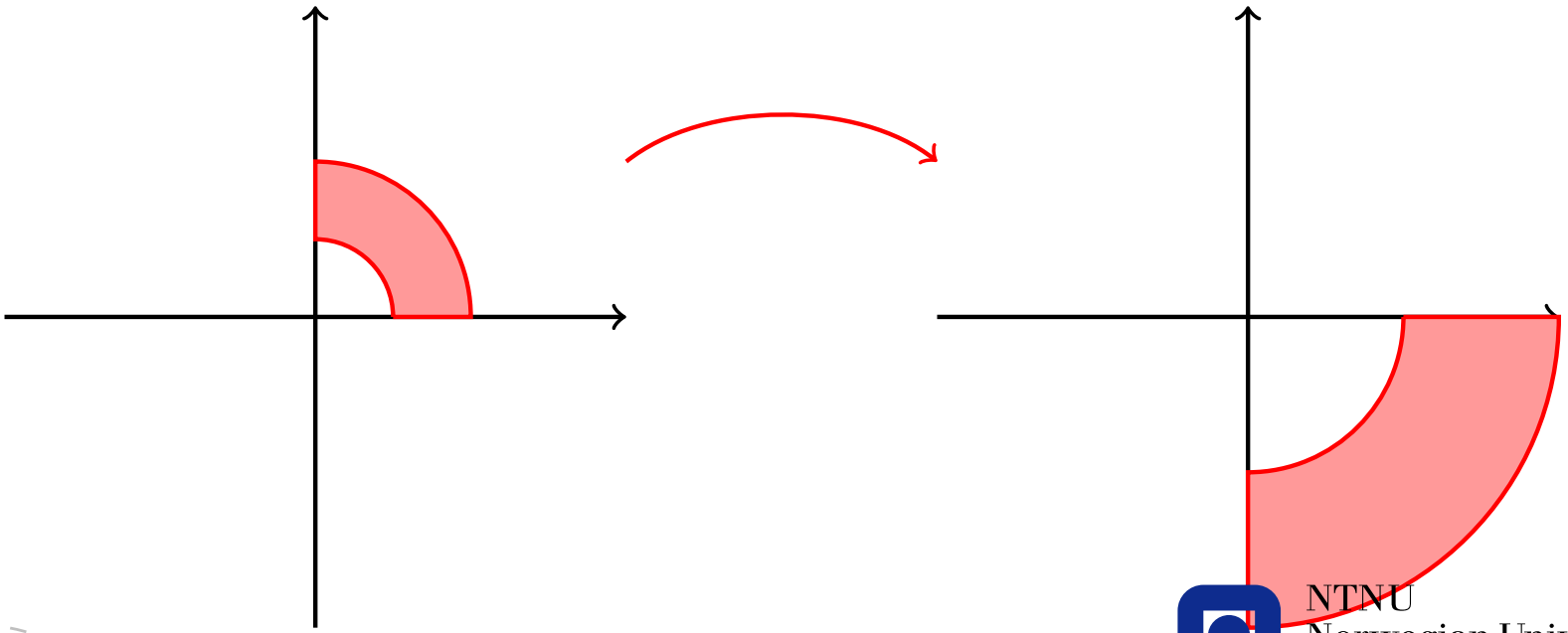
# Graphic representations of complex functions

- We cannot draw the graph of a complex function since we would need 4 dimensions to do that.
- Instead, we can graphically represent the behavior of a complex function $w = f(z)$ by drawing the $z$-plane and the $w$-plane separately, and showing the image in the $w$-plane of certain, appropriately chosen set of points in the $z$-plane.

NTNU
Norwegian University of
Science and Technology

# Example

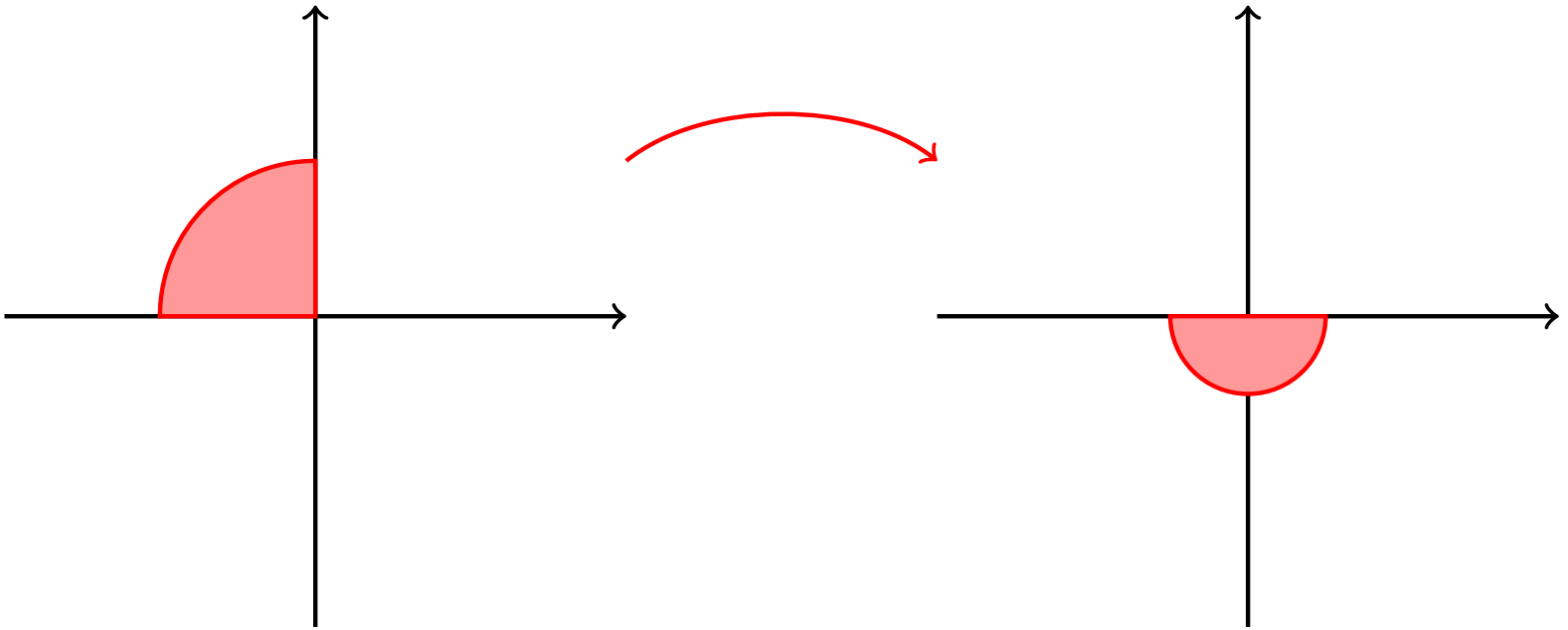
Consider the function $f(z) = -2iz$.
$\text{Arg}(-2i) = -\pi/2$ and $|-2i| = 2$, so $f$ maps the region $1/2 \leq |z| \leq 1$, $0 \leq \arg(z) \leq \pi/2$ to the region $1 \leq |z| \leq 2$, $-\pi/2 \leq \arg(z) \leq 0$.



NTNU
Norwegian University of
Science and Technology

# Example

Consider the function $g(z) = z^2$.
$g$ maps the region $0 \leq |z| \leq 1/2$, $\pi/2 \leq \arg(z) \leq \pi$ to the region $0 \leq |z| \leq 1/4$, $\pi \leq \arg(z) \leq 2\pi$.



NTNU
Norwegian University of Science and Technology

# Complex functions

Limits, continuity and differentiability of complex functions can be defined just as for real functions.

## Examples of complex functions

- Every complex polynomial is differentiable, and hence continuous.
- The functions $f(z) = \mathrm{Re}(z)$, $g(z) = \mathrm{Im}(z)$, $h(z) = |z|$ and $k(z) = \overline{z}$ are continuous, but not differentiable.
- The function $j(z) = \mathrm{Arg}(z)$ is not continuous.

NTNU
Norwegian University of
Science and Technology

# The exponential function

One can show that the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely for all complex numbers $z$.

We denote the sum of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ as the exponential function $e^z$.

$e^z$ is also the limit $\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$.

# The exponential function and cos and sin

If $y$ is a real number, then

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iy)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{(2n+1)!} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} = \cos(y) + i \sin(y).$$

NTNU
Norwegian University of
Science and Technology

# The exponential function

One can show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$. It follows that $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$ for $z = x + iy$.

# The exponential function and polar representation

If $z \neq 0$, then

$$\begin{aligned} z &= |z|(\cos(\arg(z)) + i\sin(\arg(z))) \\ &= e^{\ln(|z|)} e^{i\arg(z)} = e^{\ln(|z|)+i\arg(z)}. \end{aligned}$$

The exponential function is not injective (because $e^{x+iy} = e^{x+i(y+2\pi)}$), and does therefore not have an inverse.

NTNU
Norwegian University of
Science and Technology

# Properties of the exponential function

If $z = x + iy$, then

- $\overline{e^z} = e^{\bar{z}}$
- $\mathrm{Re}(e^z) = e^x \cos y$
- $\mathrm{Im}(e^z) = e^x \sin y$
- $|e^z| = e^x$
- $\arg(e^z) = y$

One can also show that $e^z$ is differentiable and that $\frac{d}{dz} e^z = e^z$.

NTNU
Norwegian University of
Science and Technology

# Sine and cosine

One can show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ converge absolutely for all complex numbers $z$.

We denote the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ as $\cos(z)$, and the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ as $\sin(z)$.

NTNU
Norwegian University of
Science and Technology

# Properties of sin and cos

- If $z$ is a complex number, then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

- sin and cos are periodic with period $2\pi$.
- sin and cos are differentiable and $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$.

NTNU
Norwegian University of
Science and Technology

# Hyperbolic sine and cosine

One can show that the series $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ converge absolutely for all complex numbers $z$.

We denote the sum of $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ as $\cosh(z)$, and the sum of $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ as $\sinh(z)$.

NTNU
Norwegian University of
Science and Technology

# Properties of sinh and cosh

- If $z$ is a complex number, then

$$\cosh z = \frac{e^z + e^{-z}}{2} \text{ and } \sinh z = \frac{e^z - e^{-z}}{2}.$$

- sinh and cosh are periodic with period $2\pi i$.
- sinh and cosh are differentiable and $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$.

NTNU
Norwegian University of
Science and Technology

# Problem 1 from the exam from June 2010

Solve the equation $z^2 - 2iz - 1 - 2i = 0$. Write your answer in the form $x + iy$.

# Solution

The solutions are $z = \dfrac{2i \pm \sqrt{(2i)^2 - 4(-1-2i)}}{2}$, so we must compute the square roots of
$(2i)^2 - 4(-1-2i) = -4 + 4 + 8i = 8i$.
Since $|8i| = 8$ and $\operatorname{Arg}(8i) = \frac{\pi}{2}$, the two square roots of $8i$ are

$$w_0 = \sqrt{8}\big(\cos(\pi/4) + i\sin(\pi/4)\big) = \sqrt{8}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$= 2 + 2i$$

$$w_1 = \sqrt{8}\big(\cos(5\pi/4) + i\sin(5\pi/4)\big) = \sqrt{8}\left(\frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}\right)$$
$$= -2 - 2i$$

NTNU
Norwegian University of
Science and Technology

# Solution

So the two solutions of $z^2 - 2iz - 1 - 2i = 0$ are

$$z = \frac{2i \pm \sqrt{(2i)^2 - 4(-1-2i)}}{2} = \frac{2i \pm \sqrt{8i}}{2}$$

$$= \frac{2i \pm (2+2i)}{2} = \begin{cases} 1+2i \\ -1. \end{cases}$$

NTNU
Norwegian University of
Science and Technology

# Problem 1 from the exam from August 2011

Find all complex numbers $z$ such that $\mathrm{Im}(-z+i) = (z+i)^2$. Draw the solutions on a diagram.

NTNU
Norwegian University of Science and Technology

# Solution

Let $z = x + iy$ where $x$ and $y$ are real numbers. Then $\mathrm{Im}(-z + i) = \mathrm{Im}(-x - iy + i) = 1 - y$, and $(z + i)^2 = (x + iy + i)^2 = x^2 - (y + 1)^2 + 2x(y + 1)i$.
So $\mathrm{Im}(-z + i) = (z + i)^2$ if and only if $1 - y = x^2 - (y + 1)^2$ and $x(y + 1) = 0$. If $x(y + 1) = 0$, then either $x = 0$ or $y = -1$.
Assume $x = 0$. Then $x^2 - (y + 1)^2 = -(y + 1)^2$. So $1 - y = x^2 - (y + 1)^2$ if and only if $y^2 + y + 2 = 0$, but $y^2 + y + 2 = (y + 1/2)^2 + 7/4 > 0$ for all real numbers $y$. So $1 - y$ cannot be equal to $x^2 - (y + 1)^2$ if $x = 0$.

NTNU
Norwegian University of
Science and Technology

# Solution

Assume then that $y = -1$. Then $1 - y = 2$ and $x^2 - (y+1)^2 = x^2$. So $1 - y = x^2 - (y+1)^2$ if and only if $x = \pm\sqrt{2}$.
It follows that $\mathrm{Im}(-z + i) = (z + i)^2$ if and only if $z = \pm\sqrt{2} - i$.

# Plan for tomorrow

Tomorrow we shall

- study second-order linear differential equations,
- introduce the *Wronskian*,
- completely solve second-order homogeneous linear differential equations with constant coefficients.

Section 4.1 and 4.3 in “Second-Order Equations” (pages xxxv-xlv and xlix-lv).

NTNU
Norwegian University of
Science and Technology

# Second-order linear differential equations

A *second-order linear differential equation* is a differential equation with can be written on the form

$$y'' + p(t)y' + q(t)y = g(t).$$

Such an equation is *homogeneous* if $g(t) = 0$.