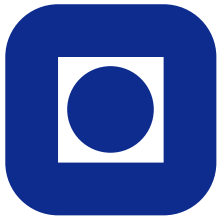

NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 28, April 25

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Yesterday's lecture

Yesterday we looked at

- *least-squares problems*,
- applications to linear models.

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

Symmetric matrices

- A *symmetric* matrix is a matrix A such that $A^T = A$.
- A symmetric matrix is necessarily square.
- We will in this lecture see that every symmetric matrix is orthogonally diagonalizable, that is, if A is symmetric, then $A = PDP^{-1}$ where P is an orthogonal matrix and D is a diagonal matrix.

NTNU
Norwegian University of
Science and Technology

Examples of symmetric matrices

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -3 \\ 1 & 2 & 4 \\ -3 & 4 & 7 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Orthogonal matrices

Recall that an *orthogonal matrix* is a square matrix whose columns form an orthonormal set.

Theorem

Let P be an $n \times n$ matrix. Then the following statements are logically equivalent:

1. P is an orthogonal matrix.
2. The columns of P form an orthonormal basis for $\mathbb{R}^n$.
3. $P^T P = I_n$.
4. P is invertible and $P^{-1} = P^T$.
5. $PP^T = I_n$.
6. The rows of P form an orthonormal basis for $\mathbb{R}^n$.
7. P^T is an orthogonal matrix.

NTNU
Norwegian University of Science and Technology

Example

Let $P = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}$.

Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set, so P is an orthogonal matrix. Notice also that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

We have that

$$P^T P = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \mathbf{u}_1 \cdot \mathbf{u}_3 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \mathbf{u}_2 \cdot \mathbf{u}_3 \\ \mathbf{u}_3 \cdot \mathbf{u}_1 & \mathbf{u}_3 \cdot \mathbf{u}_2 & \mathbf{u}_3 \cdot \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

It follows (from the inverse matrix theorem) that P is invertible and $P^{-1} = P^T$. So $P^T P = I_3$, which means that the rows of P forms an orthonormal basis for $\mathbb{R}^3$ and that P^T is orthogonal.

NTNU
Norwegian University of
Science and Technology

Orthogonally diagonalization

An $n \times n$ matrix A is *orthogonally diagonalizable* if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Theorem 2

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric.

NTNU
Norwegian University of
Science and Technology

Example

Let $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Let us find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution

The eigenvalues of A are -2 and 7.

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ form a basis for the eigenspace of A corresponding to 7.

$\mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to -2.

Notice that $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

Let $\mathbf{z}_1 = \mathbf{v}_1$ and $\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{z}_1}{\mathbf{z}_1 \cdot \mathbf{z}_1}\mathbf{z}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix}$.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

And let $\mathbf{z}_3 = \mathbf{v}_3$. Then $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is an orthogonal set. Let

$$\mathbf{x}_1 = \frac{1}{\|\mathbf{z}_1\|}\mathbf{z}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\mathbf{x}_2 = \frac{1}{\|\mathbf{z}_2\|}\mathbf{z}_2 = \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \text{ and}$$

$$\mathbf{x}_3 = \frac{1}{\|\mathbf{z}_3\|}\mathbf{z}_3 = \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal set, so if we let $P = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}$, then P is an orthogonal matrix, and if we let $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then $A = PDP^{-1}$.

NTNU
Norwegian University of
Science and Technology

Orthogonal eigenvectors

Theorem 1

If A is a symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 1

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively, and assume that $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned}\lambda_1\mathbf{v}_1 \cdot \mathbf{v}_2 &= (\lambda_1\mathbf{v}_1)^T\mathbf{v}_2 = (A\mathbf{v}_1)^T\mathbf{v}_2 = \mathbf{v}_1^T A^T\mathbf{v}_2 \\ &= \mathbf{v}_1^T A\mathbf{v}_2 = \mathbf{v}_1^T\lambda_2\mathbf{v}_2 = \mathbf{v}_1 \cdot (\lambda_2\mathbf{v}_2) = \lambda_2\mathbf{v}_1 \cdot \mathbf{v}_2\end{aligned}$$

so $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Since $\lambda_1 \neq \lambda_2$, it follows that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

The spectral theorem for symmetric matrices

Theorem 3

An $n \times n$ symmetric matrix A has the following properties:

1. A has n real eigenvalues, counting multiplicities.
2. The dimensions of the eigenspace for each eigenvalue λ equals the multiplicity of λ.
3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

NTNU
Norwegian University of Science and Technology

Spectral decomposition

Suppose $A = PDP^{-1}$ where $P = [\mathbf{u}_1 \dots \mathbf{u}_n]$ is an orthogonal matrix and $D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$.

If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$ and $A\mathbf{x} = \lambda_1(\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + \lambda_n(\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$.

We furthermore have that

$$A = \lambda \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

This is called a *spectral decomposition* of A.

NTNU
Norwegian University of
Science and Technology

Example

If $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, then

$$A = 7 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} + 7 \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{bmatrix}$$

$$- 2 \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= 7 \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + 7 \begin{bmatrix} \frac{1}{18} & \frac{-4}{18} & \frac{-1}{18} \\ \frac{-4}{18} & \frac{16}{18} & \frac{4}{18} \\ \frac{-1}{18} & \frac{4}{18} & \frac{1}{18} \end{bmatrix} - 2 \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{-2}{9} & \frac{4}{9} \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Quadratic forms

- A *quadratic form* on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric $n \times n$ matrix A.
- The matrix A is called the *matrix of* Q.
- Quadratic forms occupy a central place in various branches of mathematics, including
  - number theory,
  - linear algebra,
  - group theory,
  - differential geometry,
  - differential topology,
  - Lie theory.

NTNU
Norwegian University of
Science and Technology

Examples of quadratic forms

- $\mathbf{x} \to \|\mathbf{x}\|^2$ is a quadratic form because $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x}^T I_n \mathbf{x}$.
- $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$ is a quadratic from because
  $$6x_1^2 - 24x_1x_2 - x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
  Let $y_1 = \frac{4}{5}x_1 - \frac{3}{5}x_2$ and $y_2 = \frac{3}{5}x_1 + \frac{4}{5}x_2$. Then $6x_1^2 - 24x_1x_2 - x_2^2 = 15y_1^2 - 10y_2^2$.

NTNU
Norwegian University of
Science and Technology

The principal axes theorem

Theorem 4

Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product term.

The above theorem can be used to classify conic sections.

NTNU
Norwegian University of
Science and Technology

Example

Consider the quadratic form $(x_1, x_2) \mapsto 6x_1^2 - 24x_1x_2 - x_2^2$.

The matrix of this quadratic form is $A = \begin{bmatrix} 6 & -12 \\ -12 & -1 \end{bmatrix}$.

$\det(A - \lambda I_2) = \begin{vmatrix} 6-\lambda & -12 \\ -12 & -1-\lambda \end{vmatrix} = (6-\lambda)(-1-\lambda) - 144 = \lambda^2 - 5\lambda - 150$. and the zeros of $\lambda^2 - 5\lambda - 150$ are $\lambda = \frac{5 \pm \sqrt{5^2 - 4(-150)}}{2} = \frac{5 \pm \sqrt{625}}{2} = \frac{5 \pm 25}{2} = \begin{cases} 15 \\ -10 \end{cases}$, so the eigenvalues of A are 15 and -10.

NTNU
Norwegian University of
Science and Technology

Example (cont.)

$A - 15I_2 = \begin{bmatrix} -9 & -12 \\ -12 & -16 \end{bmatrix}$ so $\mathbf{v}_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue 15.

$A + 10I_2 = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}$ so $\mathbf{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvalue of A corresponding to the eigenvalue -10.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 5$, so if we let $P = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix}$, then P is an orthogonal matrix and $A = PDP^{-1}$.

NTNU
Norwegian University of
Science and Technology

Example (cont.)

Let $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5}x_1 + \frac{3}{5}x_2 \\ \frac{-3}{5}x_1 + \frac{4}{5}x_2 \end{bmatrix}$.

Then

$$\begin{aligned} 6x_1^2 - 24x_1x_2 - x_2^2 &= [x_1 \ x_2]A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2]P^{-1}DP \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= [y_1 \ y_2]D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [y_1 \ y_2] \begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= 15y_1^2 - 10y_2^2 \end{aligned}$$

NTNU
Norwegian University of
Science and Technology

Example (cont.)

The set of points (y_1, y_2) satisfying $15y_1^2 - 10y_2^2 = 1$ is a hyperbola. It follows that the set of points (x_1, x_2) satisfying $6x_1^2 - 24x_1x_2 - x_2^2 = 1$ is a rotated hyperbola.

NTNU
Norwegian University of
Science and Technology

Classifying quadratic forms

Definition

A quadratic form Q is said to be:

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- *indefinite* if $Q(\mathbf{x})$ assumes both positive and negative values.

- If Q is a positive definite quadratic form on $\mathbb{R}^2$, then set of points (x_1, x_2) satisfying $Q(x_1, x_2) = 1$ forms an ellipse.
- If Q is an indefinite quadratic form on $\mathbb{R}^2$, then set of points (x_1, x_2) satisfying $Q(x_1, x_2) = 1$ forms a hyperbola.

NTNU
Norwegian University of
Science and Technology

Quadratic forms and eigenvalues

Theorem 5

Let A be an $n \times n$ symmetric matrix, and let Q be the quadratic form $x \mapsto \mathbf{x}^t A\mathbf{x}$.

1. Q is positive definite if and only if the eigenvalues of A are all positive.
2. Q is negative definite if and only if the eigenvalues of A are all negative.
3. Q is indefinite if and only if A has both positive and negative eigenvalues.

NTNU
Norwegian University of
Science and Technology

Problem 7 from June 2010

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of A.
2. Find a matrix P and a diagonal matrix D such that $A = PDP^T$.
3. Solve the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 + y_2 + y_3 \\ y_2' &= y_1 + 2y_2 \\ y_3' &= y_1 + 2y_3 \end{aligned}$$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$.

NTNU
Norwegian University of
Science and Technology

Solution

$$
\begin{aligned}
\det(A-\lambda I_3) &= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix} + (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\
&= -(2-\lambda) + (2-\lambda)\big((3-\lambda)(2-\lambda)-1\big) \\
&= (2-\lambda)\big((3-\lambda)(2-\lambda)-2\big) \\
&= (2-\lambda)(\lambda^2-5\lambda-4)
\end{aligned}
$$

and $\lambda^2-5\lambda-4=0$ if and only if

$$
\lambda = \frac{5\pm\sqrt{5^2-4(-4)}}{2} = \frac{5\pm 3^2}{2} = \begin{cases} 4 \\ 1 \end{cases},
$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

so the eigenvalues of A are 1,2 and 4.
To find the eigenvectors of A corresponding to 1, we reduce $A - I_3$ to its reduced echelon form.

$$A - I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set of eigenvectors of A corresponding to 1 is $\left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$A - 2I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ so the set of eigenvectors of A corresponding to 2 is $$\left\{ t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : t \neq 0 \right\}.$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

To find the eigenvectors of A corresponding to 4, we reduce $A - 4I_3$ to its reduced echelon form.

$$A - 4I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set of eigenvectors of A corresponding to 4 is $\left\{ t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$.

Solution (cont.)

To find a matrix P and a diagonal matrix D such that $A = PDP^T$ we will orthogonally diagonalize A. For that we need an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Since A is symmetric, eigenvectors of A corresponding to different eigenvalues are orthogonal to each other, so we just have to find a unit vector in each of the 3 eigenspaces of A.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

We have that $\begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector of A corresponding to 1, that $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ is a unit eigenvector of A corresponding to 2, and that $\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector of A corresponding to 4.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

So if we let $P = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$A = PDP^{-1} = PDP^{T}$.

Solution (cont.)

The coefficient matrix the system

$$\begin{aligned} y_1' &= 3y_1 + y_2 + y_3 \\ y_2' &= y_1 + 2y_2 \\ y_3' &= y_1 + 2y_3 \end{aligned}$$

is A,

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

so since the eigenvalues of A are 1,2 and 4, and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 1, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to 2, and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 4, the general solution of the system is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t}.$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\text{If } \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t}, \text{ then}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

so $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$ if and only if

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

We reduce the augmented matrix of the above system to its reduced echelon form.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

$$\begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We see that the solution to the system

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

is $c_1 = -1, c_2 = 2, c_3 = 1$.

NTNU
Norwegian University of
Science and Technology

Solution (cont.)

So the solution to the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 + y_2 + y_3 \\ y_2' &= y_1 + 2y_2 \\ y_3' &= y_1 + 2y_3 \end{aligned}$$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$, is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t + 2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} 2e^{4t} + e^t \\ e^{4t} + 2e^{2t} - e^t \\ e^{4t} - 2e^{2t} - e^t \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology