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TMA4115 - Calculus 3

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Yesterday's lecture

Yesterday we introduced and studied

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- the *orthogonal complement* of a subspace,
- *orthogonal bases* and *orthonormal bases*,
- *orthogonal* matrices.

Today's lecture

Today we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.
$W^\perp$ is a subspace of $\mathbb{R}^n$.

The orthogonal complements of Row(A) and Col(A)

Theorem 3

Let A be an $m \times n$ matrix. Then

1. $(\mathrm{Row}(A))^{\perp} = \mathrm{Nul}(A)$.
2. $(\mathrm{Col}(A))^{\perp} = \mathrm{Nul}(A^T)$.

Orthogonal sets and bases

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Theorem 5

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in W. Then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

Orthonormal sets and bases

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.
- If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and $\mathbf{y}$ is in W, then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \mathbf{y} \cdot \mathbf{v}_j$.

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Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
2. $\|U\mathbf{x}\| = \|\mathbf{x}\|$.
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

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Orthogonal matrices

A square matrix with **orthonormal** columns is called an *orthogonal matrix*.

A square matrix U is orthogonal if and only if U is invertible and $U^{-1} = U^T$.

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Orthogonal projections

- Let $\mathbf{u}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector $\text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ onto L (or onto $\mathbf{u}$).
- If $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$, then $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{y} = \text{proj}_L \mathbf{y} + \mathbf{z}$.
- The vector $\mathbf{z} = \mathbf{y} - \text{proj}_L \mathbf{y}$ is called the component of $\mathbf{y}$ orthogonal to L (or to $\mathbf{u}$).
- $\|z\|$ is called the *distance from $\mathbf{y}$ to L*.

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Example

Let $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let us compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

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Solution

The line through $\mathbf{u}$ and the origin is the line $L = \text{Span}\{\mathbf{u}\}$.

$$\text{proj}_L\, \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$$

so the distance from $\mathbf{y}$ to L is

$$\|\mathbf{y} - \text{proj}_L\, \mathbf{y}\| = \left\| \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} \right\|$$
$$= \sqrt{(3/2)^2 + (-3/2)^2} = \sqrt{9/2} = \frac{3}{\sqrt{2}}.$$

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The orthogonal decomposition theorem

Theorem 8

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\mathbf{y}$ can be written uniquely in the form $\mathbf{y} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.
2. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W, then $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ and $\mathbf{z} = \mathbf{y} - \mathbf{w}$.

The vector $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$ is called the *orthogonal projection of* $\mathbf{y}$ *onto* W and is denoted by $\text{proj}_W\,\mathbf{y}$.

Proof of Theorem 8

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let $\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$.
Then $\mathbf{w}$ is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then $\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$ for each k, so $\mathbf{z}$ is in $W^{\perp}$.
Suppose $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$ where $\mathbf{w}'$ is in W and $\mathbf{z}'$ is in $W^{\perp}$. Then $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z}$ is both in W and in $W^{\perp}$, so $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z} = \mathbf{0}$, from which it follows that $\mathbf{w} = \mathbf{w}'$ and $\mathbf{z} = \mathbf{z}'$.

The best approximation theorem

Theorem 9

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be in $\mathbb{R}^n$, and let $\mathbf{w} = \text{proj}_W \mathbf{y}$.
Then $\mathbf{w}$ is the closest point in W to $\mathbf{y}$ in the sense that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}$ in W distinct from $\mathbf{w}$.

Proof of Theorem 9

Let $\mathbf{x}$ be a vector in W distinct from $\mathbf{w}$. Then $\mathbf{w} - \mathbf{x}$ is in W and $\mathbf{y} - \mathbf{w}$ is in $W^\perp$, so $\mathbf{w} - \mathbf{x}$ and $\mathbf{y} - \mathbf{w}$ are orthogonal. It follows that

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|(\mathbf{y} - \mathbf{w}) + (\mathbf{w} - \mathbf{x})\|^2 = \|\mathbf{y} - \mathbf{w}\|^2 + \|\mathbf{w} - \mathbf{x}\|^2.$$

Since $\mathbf{x} \neq \mathbf{w}$, it follows that $\|\mathbf{w} - \mathbf{x}\| > 0$, and thus that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$.

Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us find the closest point in W to $\mathbf{y}$.

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Solution

The closest point in W to $\mathbf{y}$ is $\text{proj}_W \mathbf{y}$. We have that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W. It follows that

$$\begin{aligned}
\text{proj}_W \mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\
&= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.
\end{aligned}$$

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Orthogonal projections onto orthonormal bases

Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in $\mathbb{R}^n$.

1. Then $\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$.
2. If $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$, then $\text{proj}_W \mathbf{y} = UU^T\mathbf{y}$.

Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W. Then $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for each k, so
$\text{proj}_W \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p$
for each $\mathbf{y}$ in $\mathbb{R}^n$.
Let $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$. Then

$$UU^T\mathbf{y} = [\mathbf{u}_1 \ldots \mathbf{u}_p][\mathbf{u}_1 \ldots \mathbf{u}_p]^T\mathbf{y} = [\mathbf{u}_1 \ldots \mathbf{u}_p]\begin{bmatrix}\mathbf{u}_1\cdot\mathbf{y}\\ \vdots \\ \mathbf{u}_p\cdot\mathbf{y}\end{bmatrix}$$
$$= (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p = \text{proj}_W \mathbf{y}$$

for each $\mathbf{y}$ in $\mathbb{R}^n$.

Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us use Theorem 10 to find the closest point in W to $\mathbf{y}$.

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Solution

$\|\mathbf{u}_1\|^2 = \mathbf{u}_1 \cdot \mathbf{u}_1 = 30$ and $\|\mathbf{u}_2\|^2 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 6$, so

$\left\{ \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for W.

Let $U = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}$.

Solution

Then

$$UU^T = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$

so $\text{proj}_W \mathbf{y} = UU^T\mathbf{y} = \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$

An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
3. If appropriate, scale $\mathbf{v}_2$ to simplify later calculations.
4. Let $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$, let $W_3 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and scale $\mathbf{v}_3$ to simplify later calculations (if appropriate).

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An algorithm for producing an orthogonal basis for a subspace of $\mathbb{R}^n$

5. Continue like this and produce vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ where, for $1 < k \leq p$, $\mathbf{v}_k$ is an appropriate multiple of $\mathbf{x}_k - \text{proj}_{\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k - \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\mathbf{v}_{k-1}$.
6. Then $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is an orthogonal basis for W.

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The Gram-Schmidt process

Theorem 11

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a linearly independent subset of $\mathbb{R}^n$. Let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_k - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set, and $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $1 \leq k \leq p$.

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Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1.

Suppose that $1 \leq k < p$ and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W_k. Then $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$ is orthogonal to W_k and in W_{k+1}, so $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}$ is an orthogonal set in W_{k+1}. Since $\dim(W_{k+1}) = k + 1$, it follows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}$ is an orthogonal basis for W_{k+1}.

It follows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W_k for all $1 \leq k \leq p$.

Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and
$W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.
Let us find an orthogonal basis for W.

Solution

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$.
Let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}\mathbf{x}_2 = \mathbf{x}_2 - \tfrac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \tfrac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \tfrac{1}{4}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},$$

$$\mathbf{v}_2' = 4\mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \text{ and}$$

$W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2'\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

Solution

Let

$$
\begin{aligned}
\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{W_2} \mathbf{x}_3 \\
&= x_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2' \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

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Solution (cont.)

and $\mathbf{v}_3' = 3\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$ is an orthogonal basis

for W.

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The QR factorization

Theorem 12

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where

- Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col(A),
- R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

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Proof of Theorem 12

Let $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$ and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthogonal set such that $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \leq k \leq p$.

Then we have for each $1 \leq k \leq p$ that there exist scalars $r_{1k}, r_{2k}, \dots, r_{kk}$ such that $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \dots r_{kk}\mathbf{v}_k$.

We must have that $r_{kk} \neq 0$ because otherwise $\mathbf{x}_k$ would be in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k_1}\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_{k_1}\}$ which would contradict the assumption that $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is linearly independent.

We may assume that $r_{kk} > 0$, because if $r_{kk} < 0$, then we can replace $\mathbf{v}_k$ by $-\mathbf{v}_k$ and r_{kk} by $-r_{kk}$, and then $r_{kk} > 0$.

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Proof of Theorem 12 (cont.)

Let $Q = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$.

Then $QR = [\mathbf{x}_1 \dots \mathbf{x}_n] = A$.

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Example

Let us find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

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Solution

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$\mathbf{v}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$.

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Solution (cont.)

Then $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set such that

$$\begin{aligned}
\mathbf{x}_1 &= 2\mathbf{v}_1 \\
\mathbf{x}_2 &= \frac{3}{2}\mathbf{v}_1 + \frac{\sqrt{3}}{2}\mathbf{v}_2 \\
\mathbf{x}_3 &= \mathbf{v}_1 + \frac{1}{\sqrt{3}}\mathbf{v}_2 + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{v}_3
\end{aligned}$$

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Solution (cont.)

So if we let $Q = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and

$R = \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$, then $QR = A$ is a QR factorization of A.

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Problem 6 from June 2010

Let A be the following matrix; find a basis for each of the spaces Nul(A), Col(A), $(\mathrm{Col}(A))^\perp$, and Row(A).

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix}$$

Find the orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to Col(A).

Solution

We start by reducing A to its reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 2 & -6 & -8 & -8 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

We see that $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Nul}(A)$, that $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}(A)$,

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Solution (cont.)

and that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ -4 \\ -4 \end{bmatrix} \right\}$ is a basis for Row(A).

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Solution (cont.)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $(\mathrm{Col}(A))^\perp$ if and only if

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = x_1 + 3x_2 + 4x_3 = 0$ and

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = x_2 + 2x_3 = 0.$

We reduce the coefficient matrix of the system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

to its reduced echelon form.

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Solution (cont.)

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

We see that $(\text{Col}(A))^\perp = \text{Span}\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$, so $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $(\text{Col}(A))^\perp$.

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Solution (cont.)

The orthogonal projection of $\begin{bmatrix}0\\0\\1\end{bmatrix}$ on to $(\mathrm{Col}(A))^\perp$ is

$$\frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}2\\-2\\1\end{bmatrix}}{\begin{bmatrix}2\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}2\\-2\\1\end{bmatrix}}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \tfrac{1}{9}\begin{bmatrix}2\\-2\\1\end{bmatrix}$$ so the orthogonal projection of

$\begin{bmatrix}0\\0\\1\end{bmatrix}$ on to $\mathrm{Col}(A)$ is $\begin{bmatrix}0\\0\\1\end{bmatrix} - \tfrac{1}{9}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \tfrac{1}{9}\begin{bmatrix}-2\\2\\8\end{bmatrix}$.

Problem 5 from June 2011

Let V be the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

and let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}.$$

Find the nearest point in V to $\mathbf{b}$ (that is, the orthogonal projection of $\mathbf{b}$ on to V).

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Solution

We start by reducing the matrix to an echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for V. We then find an orthogonal basis for V by using the Gram-Schmidt process on $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Solution (cont.)

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$ is an orthogonal basis for V.

Solution (cont.)

We then have that

$$\operatorname{proj}_V \mathbf{b} = \frac{\begin{bmatrix}1\\7\\3\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-1\end{bmatrix}}{\begin{bmatrix}1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-1\end{bmatrix}} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{\begin{bmatrix}1\\7\\3\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\0\end{bmatrix}}{\begin{bmatrix}2\\-1\\0\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\0\end{bmatrix}} \begin{bmatrix}2\\-1\\0\end{bmatrix}$$

$$= \frac{12}{6} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{-5}{5} \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\5\\-2\end{bmatrix}.$$

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Problem 5 from December 2010

Let $V \subseteq \mathbb{R}^4$ be the solution space of the linear system

$$x + y - z + w = 0$$
$$x + 2y - 2z + w = 0$$

1. Find an orthogonal basis for V.
2. Find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ on to V.
3. Find an orthogonal basis for $\mathbb{R}^4$ in which the first two first basis vectors are the once we found in (1).

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Solution

We start by reducing the coefficient matrix of the system to its reduced echelon form.

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

We see that $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for V and that it is orthogonal.

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Solution (cont.)

The orthogonal projection of **b** on to V is

$$\frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}+\frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

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Solution (cont.)

Let A be the coefficient matrix of the system. Then

$$V^{\perp} = (\mathrm{Nul}(A))^{\perp} = \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}, \text{ so}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ is an orthogonal basis for } \mathbb{R}^4.$$

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Plan for next week

Wednesday we shall look at

- *least-squares problems*,
- applications to linear models.

Sections 6.5–6.6 in “Linear Algebras and Its Applications” (pages 360–375).

Thursday we shall introduce and study

- *symmetric matrices*,
- *quadratic forms*.

Sections 7.1–7.2 in “Linear Algebras and Its Applications” (pages 393–407).