

TMA4115 - Calculus 3 Lecture 26, April 18

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Yesterday's lecture

Yesterday we introduced and studied

- the inner product,
- the *length* of a vector,
- orthogonality and orthogonal sets in \mathbb{R}^n ,
- the orthogonal complement of a subspace,
- orthogonal bases and orthonormal bases,
- orthogonal matrices.



Today's lecture

Today we shall introduce and study

- orthogonal projections,
- the Gram-Schmidt process,
- QR factorization.



Orthogonal complements

Let *W* be a subspace of \mathbb{R}^n . The set of all vectors **z** that are orthogonal to *W* is called the *orthogonal complement* of *W* and is denoted by W^{\perp} . W^{\perp} is a subspace of \mathbb{R}^n .



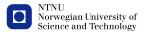
The orthogonal complements of Row(A) and Col(A)

Theorem 3

Let *A* be an $m \times n$ matrix. Then

$$(\operatorname{Row}(A))^{\perp} = \operatorname{Nul}(A)$$

$$(\operatorname{Col}(A))^{\perp} = \operatorname{Nul}(A^{T}).$$



Orthogonal sets and bases

- A set of vectors {v₁,..., v_p} in ℝⁿ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if v_i · v_j = 0 whenever i ≠ j.
- An orthogonal basis for a subspace W of ℝⁿ is a basis for W that is also an orthogonal set.
- { $\mathbf{v}_1, \ldots, \mathbf{v}_p$ } is an orthogonal basis for W if and only if { $\mathbf{v}_1, \ldots, \mathbf{v}_p$ } is an orthogonal set and Span{ $\mathbf{v}_1, \ldots, \mathbf{v}_p$ } = W.

Theorem 5

Let $\mathcal{B} = {\mathbf{v}_1, \dots, \mathbf{v}_p}$ be an orthogonal basis for a subspace W of \mathbb{R}^n and let \mathbf{y} be in W. Then the coordinates c_1, \dots, c_p of \mathbf{y} relative to \mathcal{B} is given by $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

Orthonormal sets and bases

- An orthonormal set is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthonormal set if and only if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- An orthonormal basis for a subspace W of \mathbb{R}^n is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = W$.
- If B = {v₁,..., v_p} be an orthonormal basis for a subspace W of ℝⁿ and y is in W, then the coordinates c₁,..., c_p of y relative to B is given by c_j = y ⋅ v_j.



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Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^{T}U = I_{n}$.

Theorem 7

Let *U* be an $m \times n$ matrix with orthonormal columns and let **x** and **y** be in \mathbb{R}^n . Then

$$\bigcirc (U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}.$$

$$\| U \mathbf{x} \| = \| \mathbf{x} \|.$$

3 $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.



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Orthogonal matrices

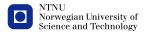
A square matrix with **orthonormal** columns is called an *orthogonal matrix*.

A square matrix U is orthogonal if and only if U is invertible and $U^{-1} = U^{T}$.



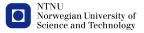
Orthogonal projections

- Let **u** and **y** be vectors in \mathbb{R}^n and assume that $\mathbf{u} \neq \mathbf{0}$.
- Let $L = \text{Span}\{\mathbf{u}\}$.
- The vector proj_L y = ^{y·u}/_{u·u} u is called the *orthogonal* projection of y onto L (or onto u).
- If z = y proj_L y, then z is orthogonal to u and y = proj_L y + z.
- The vector z = y proj_L y is called the component of y orthogonal to L (or to u).
- ||z|| is called the *distance from* **y** to *L*.



Example

Let $\mathbf{y} = \begin{bmatrix} 5\\2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1\\1 \end{bmatrix}$. Let us compute the distance from \mathbf{y} to the line through \mathbf{u} and the origin.



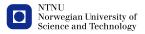
Solution

The line through **u** and the origin is the line $L = \text{Span}\{\mathbf{u}\}$.

$$\operatorname{proj}_{L} \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{7}{2} \begin{bmatrix} 1\\1 \end{bmatrix} = \begin{bmatrix} 7/2\\7/2 \end{bmatrix}$$

so the distance from **y** to *L* is

$$\|\mathbf{y} - \operatorname{proj}_{L} \mathbf{y}\| = \left\| \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} \right\|$$
$$= \sqrt{(3/2)^{2} + (-3/2)^{2}} = \sqrt{9/2} = \frac{3}{\sqrt{2}}.$$



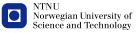
The orthogonal decomposition theorem

Theorem 8

Let *W* be a subspace of \mathbb{R}^n and let **y** be in \mathbb{R}^n .

- Then y can be written uniquely in the form y = w + zwhere w is in W and z is in W^{\perp} .
- If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for W, then $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$ and $\mathbf{z} = \mathbf{y} - \mathbf{w}$.

The vector $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$ is called the *orthogonal* projection of \mathbf{y} onto W and is denoted by $\operatorname{proj}_W \mathbf{y}$.



Proof of Theorem 8

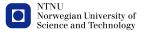
Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W and let $\mathbf{w} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$. Then \mathbf{w} is in W. Let $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then $\mathbf{z} \cdot \mathbf{u}_k = \mathbf{y} \cdot \mathbf{u}_k - \mathbf{y} \cdot \mathbf{u}_k = 0$ for each k, so \mathbf{z} is in W^{\perp} . Suppose $\mathbf{y} = \mathbf{w}' + \mathbf{z}'$ where \mathbf{w}' is in W and \mathbf{z}' is in W^{\perp} . Then $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z}$ is both in W and in W^{\perp} , so $\mathbf{w} - \mathbf{w}' = \mathbf{z}' - \mathbf{z} = \mathbf{0}$, from which it follows that $\mathbf{w} = \mathbf{w}'$ and $\mathbf{z} = \mathbf{z}'$.



The best approximation theorem

Theorem 9

Let *W* be a subspace of \mathbb{R}^n , let **y** be in \mathbb{R}^n , and let $\mathbf{w} = \operatorname{proj}_W \mathbf{y}$. Then **w** is the closest point in *W* to **y** in the sense that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$ for all **x** in *W* distinct from **w**.



Proof of Theorem 9

Let **x** be a vector in *W* distinct from **w**. Then $\mathbf{w} - \mathbf{x}$ is in *W* and $\mathbf{y} - \mathbf{w}$ is in W^{\perp} , so $\mathbf{w} - \mathbf{x}$ and $\mathbf{y} - \mathbf{w}$ are orthogonal. It follows that

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|(\mathbf{y} - \mathbf{w}) + (\mathbf{w} - \mathbf{x})\|^2 = \|\mathbf{y} - \mathbf{w}\|^2 + \|\mathbf{w} - \mathbf{x}\|^2$$

Since $\mathbf{x} \neq \mathbf{w}$, it follows that $\|\mathbf{w} - \mathbf{x}\| > 0$, and thus that $\|\mathbf{y} - \mathbf{w}\| < \|\mathbf{y} - \mathbf{x}\|$.



Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2\\5\\-1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1\\2\\3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us find the closest point in W to \mathbf{y} .



Solution

The closest point in W to **y** is proj_W **y**. We have that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W. It follows that

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$$\operatorname{roj}_{W} \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_{1}}{\mathbf{u}_{1} \cdot \mathbf{u}_{1}} \mathbf{u}_{1} + \frac{\mathbf{y} \cdot \mathbf{u}_{2}}{\mathbf{u}_{2} \cdot \mathbf{u}_{2}} \mathbf{u}_{2}$$
$$= \frac{9}{30} \begin{bmatrix} 2\\5\\-1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2\\1\\1 \end{bmatrix}$$
$$= \begin{bmatrix} -2/5\\2\\1/5 \end{bmatrix}.$$



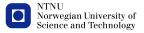
Orthogonal projections onto orthonormal bases

Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W of \mathbb{R}^n and let \mathbf{y} be in \mathbb{R}^n .

1 Then
$$\operatorname{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$
.

2 If
$$U = [\mathbf{u}_1 \dots \mathbf{u}_{\rho}]$$
, then proj_W $\mathbf{y} = UU^T \mathbf{y}$.

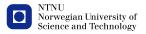


Proof of Theorem 10

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for W. Then $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for each k, so proj_W $\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$ for each \mathbf{y} in \mathbb{R}^n . Let $U = [\mathbf{u}_1 \ldots \mathbf{u}_p]$. Then

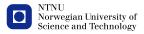
$$UU^{\mathsf{T}}\mathbf{y} = [\mathbf{u}_{1} \dots \mathbf{u}_{\rho}][\mathbf{u}_{1} \dots \mathbf{u}_{\rho}]^{\mathsf{T}}\mathbf{y} = [\mathbf{u}_{1} \dots \mathbf{u}_{\rho}]\begin{bmatrix}\mathbf{u}_{1} \cdot \mathbf{y}\\ \vdots\\ \mathbf{u}_{\rho} \cdot \mathbf{y}\end{bmatrix}$$
$$= (\mathbf{y} \cdot \mathbf{u}_{1})\mathbf{u}_{1} + \dots + (\mathbf{y} \cdot \mathbf{u}_{\rho})\mathbf{u}_{\rho} = \operatorname{proj}_{W}\mathbf{y}$$

for each **y** in \mathbb{R}^n .



Example

Let $\mathbf{u}_1 = \begin{bmatrix} 2\\5\\-1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1\\2\\3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Let us use Theorem 10 to find the closest point in W to \mathbf{y} .



Solution

$$\begin{aligned} \|\mathbf{u}_{1}\|^{2} &= \mathbf{u}_{1} \cdot \mathbf{u}_{1} = 30 \text{ and } \|\mathbf{u}_{2}\|^{2} = \mathbf{u}_{2} \cdot \mathbf{u}_{2} = 6, \text{ so} \\ \left\{ \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\} \text{ is an orthonormal basis for } W. \\ \text{Let } U &= \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}. \end{aligned}$$



Solution

Then

$$UU^{T} = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$
$$= \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}$$
so proj_W $\mathbf{y} = UU^{T} \mathbf{y} = \frac{1}{30} \begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$

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An algorithm for producing an orthogonal basis for a subspace of \mathbb{R}^n

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W of \mathbb{R}^n .

• Let
$$\mathbf{v}_1 = \mathbf{x}_1$$
 and $W_1 = \operatorname{Span}\{\mathbf{v}_1\}$.

2 Let
$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$$
 and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$

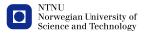
If appropriate, scale v_2 to simplify later calculations.

Let
$$\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$$
, let $W_3 = \operatorname{Span}{\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}}$, and scale \mathbf{v}_3 to simplify later calculations (if appropriate).



An algorithm for producing an orthogonal basis for a subspace of \mathbb{R}^n

- Continue like this and produce vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{\bar{p}}$ where, for $1 < k \le p$, \mathbf{v}_k is an appropriate multiple of $\mathbf{x}_k \operatorname{proj}_{\operatorname{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k \frac{\mathbf{x}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \dots \frac{\mathbf{x}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}} \mathbf{v}_{k-1}$.
- Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W.



The Gram-Schmidt process

Theorem 11

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_{\rho}\}$ be a linearly independent subset of \mathbb{R}^n . Let

$$\mathbf{v}_{1} = \mathbf{x}_{1}$$

$$\mathbf{v}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}}$$

$$\vdots$$

$$\mathbf{v}_{p} = \mathbf{x}_{k} - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \dots - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal set, and Span $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} =$ Span $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for $1 \le k \le p$.



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Proof of Theorem 11

Let $W_k = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \le k \le p$. Then $\{\mathbf{v}_1\}$ is an orthogonal basis for W_1 . Suppose that $1 \le k < p$ and that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W_k . Then $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \text{proj}_{W_k} \mathbf{x}_{k+1}$ is orthogonal to W_k and in W_{k+1} , so $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ is an orthogonal set in W_{k+1} . Since dim $(W_{k+1}) = k + 1$, it follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ is an orthogonal basis for W_{k+1} . It follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W_k for all $1 \le k \le p$.



Example

Let
$$\mathbf{x}_1 = \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix}$$
, $\mathbf{x}_2 = \begin{bmatrix} 0\\1\\1\\1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0\\0\\1\\1 \end{bmatrix}$, and
 $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.
Let us find an orthogonal basis for W .



Solution

 $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent and therefore a basis for W. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$. Let

$$\mathbf{v}_{2} = \mathbf{x}_{2} - \operatorname{proj}_{W_{1}} \mathbf{x}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} = \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_{2}' = 4\mathbf{v}_{2} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and}$$

$$W_{2} = \operatorname{Span}\{\mathbf{v}_{1}, \mathbf{v}_{2}\} = \operatorname{Span}\{\mathbf{v}_{1}, \mathbf{v}_{2}\} = \operatorname{Span}\{\mathbf{x}_{1}, \mathbf{x}_{2}\}.$$



Solution

Let

$$\mathbf{v}_{3} = \mathbf{x}_{3} - \operatorname{proj}_{W_{2}} \mathbf{x}_{3}$$

$$= x_{3} - \frac{\mathbf{x}_{3} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \frac{\mathbf{x}_{3} \cdot \mathbf{v}_{2}'}{\mathbf{v}_{2}' \cdot \mathbf{v}_{2}'} \mathbf{v}_{2}'$$

$$= \begin{bmatrix} 0\\0\\1\\1\\1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3\\1\\1\\1\\1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0\\-2\\1\\1 \end{bmatrix}$$

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Solution (cont.)

and
$$\mathbf{v}_3' = 3\mathbf{v}_3 = \begin{bmatrix} 0\\ -2\\ 1\\ 1\\ 1 \end{bmatrix}$$
. Then

$$\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\} = \left\{ \begin{bmatrix} 1\\ 1\\ 1\\ 1\\ 1 \end{bmatrix}, \begin{bmatrix} -3\\ 1\\ 1\\ 1\\ 1 \end{bmatrix}, \begin{bmatrix} 0\\ -2\\ 1\\ 1\\ 1 \end{bmatrix} \right\} \text{ is an orthogonal basis}$$
for W .



The QR factorization

Theorem 12

If *A* is an $m \times n$ matrix with linearly independent columns, then *A* can be factored as A = QR, where

- *Q* is an *m* × *n* matrix whose columns form an orthonormal basis for Col(*A*),
- *R* is an *n* × *n* upper triangular invertible matrix with positive entries on its diagonal.



Proof of Theorem 12

Let $A = [\mathbf{x}_1 \dots \mathbf{x}_n]$ and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthogonal set such that $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each $1 \le k \le p$.

Then we have for each $1 \le k \le p$ that there exist scalars

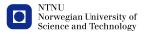
 $r_{1k}, r_{2k}, \ldots, r_{kk}$ such that $\mathbf{x}_k = r_{1k}\mathbf{v}_1 + \ldots + r_{kk}\mathbf{v}_k$.

We must have that $r_{kk} \neq 0$ because otherwise \mathbf{x}_k would be in Span{ $\mathbf{v}_1, \ldots, \mathbf{v}_{k_1}$ } = Span{ $\mathbf{x}_1, \ldots, \mathbf{x}_{k_1}$ } which would contradict the assumption that { $\mathbf{x}_1, \ldots, \mathbf{x}_n$ } is linearly independent.

We may assume that $r_{kk} > 0$, because if $r_{kk} < 0$, then we can replace \mathbf{v}_k by $-\mathbf{v}_k$ and r_{kk} by $-r_{kk}$, and then $r_{kk} > 0$.



Proof of Theorem 12 (cont.) Let $Q = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $R = \begin{bmatrix} r_{11} & r_{12} \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$. Then $QR = [\mathbf{x}_1 \dots \mathbf{x}_n] = A$.



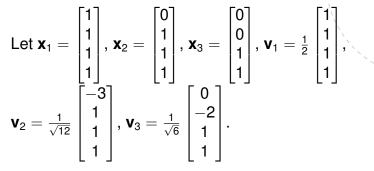
Example

Let us find a *QR* factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.



TMA4115 - Calculus 3, Lecture 26, page 35

Solution





Then $A = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set such that

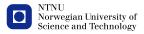
$$\begin{aligned} \mathbf{x}_1 &= 2\mathbf{v}_1 \\ \mathbf{x}_2 &= \frac{3}{2}\mathbf{v}_1 + \frac{\sqrt{3}}{2}\mathbf{v}_2 \\ \mathbf{x}_3 &= \mathbf{v}_1 + \frac{1}{\sqrt{3}}\mathbf{v}_2 + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{v}_3 \end{aligned}$$



So if we let
$$Q = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{12}}{\sqrt{12}} & 0\\ \frac{1}{2} & \frac{\sqrt{12}}{\sqrt{12}} & \frac{\sqrt{6}}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$
 and $R = \begin{bmatrix} 2 & \frac{3}{2} & 1\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}}\\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$, then $QR = A$ is a QR factorization of A .

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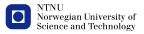


Problem 6 from June 2010

Let *A* be the following matrix; find a basis for each of the spaces Nul(*A*), Col(*A*), (Col(*A*))^{\perp}, and Row(*A*).

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix}$$

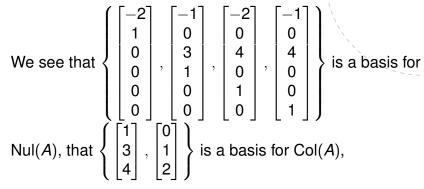
Find the orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to Col(*A*).



We start by reducing A to its reduced echelon form.

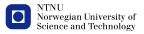
$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 2 & -6 & -8 & -8 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$







and that $\left\{ \begin{array}{c|c} 2 \\ 0 \\ 1 \\ 2 \\ 2 \\ \end{array}, \begin{array}{c} 0 \\ 1 \\ -3 \\ -4 \\ \end{array} \right\}$ is a basis for Row(A).



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is in } (\operatorname{Col}(A))^{\perp} \text{ if and only if}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = x_1 + 3x_2 + 4x_3 = 0 \text{ and}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = x_2 + 2x_3 = 0.$$
We reduce the coefficient matrix of the system

$$x_1 + 3x_2 + 4x_3 = 0$$

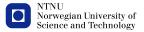
$$x_2 + 2x_3 = 0$$
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to its reduced echelon form.

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$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

We see that $(\operatorname{Col}(A))^{\perp} = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$, so $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $(\operatorname{Col}(A))^{\perp}$.



The orthogonal projection of $\begin{bmatrix} 0\\0\\1 \end{bmatrix}$ on to $(\operatorname{Col}(A))^{\perp}$ is

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Problem 5 from June 2011

Let *V* be the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

and let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$
 .

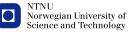
Find the nearest point in V to **b** (that is, the orthogonal projection of **b** on to V).

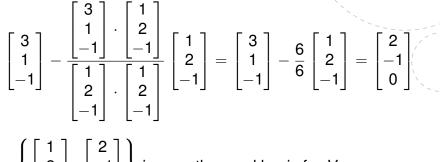


We start by reducing the matrix to an echelon form.

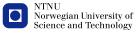
$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for *V*. We then find
an orthogonal basis for *V* by using the Gram-Schmidt
process on $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

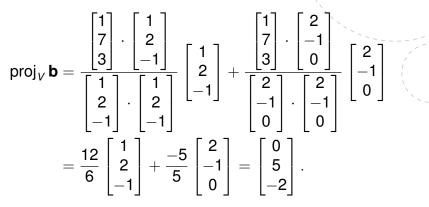


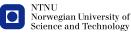


so
$$\left\{ \begin{bmatrix} 1\\ 2\\ -1 \end{bmatrix}, \begin{bmatrix} 2\\ -1\\ 0 \end{bmatrix} \right\}$$
 is an orthogonal basis for *V*.



We then have that





Problem 5 from December 2010

Let $V \subseteq \mathbb{R}^4$ be the solution space of the linear system

$$x + y - z + w = 0$$
$$x + 2y - 2z + w = 0$$



- Find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on to V.
- Sind an orthogonal basis for ℝ⁴ in which the first two first basis vectors are the once we found in (1).

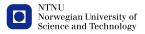


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We start by reducing the coefficient matrix of the system to its reduced echelon form.

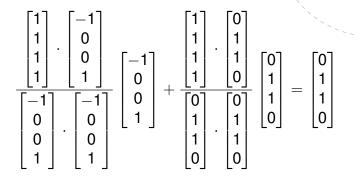
$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}^{-1}$$

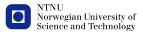
We see that $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for *V* and that it is orthogonal.



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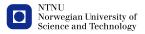
The orthogonal projection of \mathbf{b} on to V is





Let *A* be the coefficient matrix of the system. Then $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$$V^{\perp} = (\operatorname{Nul}(A))^{\perp} = \operatorname{Row}(A) = \operatorname{Span} \left\{ \begin{bmatrix} 0\\0\\1 \end{bmatrix}, \begin{bmatrix} 1\\-1\\0 \end{bmatrix} \right\}, \text{ so} \\ \left\{ \begin{bmatrix} -1\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\-1\\0 \end{bmatrix} \right\} \text{ is an orthogonal basis for } \right\}$$



Plan for next week

Wednesday we shall look at

- least-squares problems,
- applications to linear models.

Sections 6.5–6.6 in "Linear Algebras and Its Applications" (pages 360–375).

Thursday we shall introduce and study

- symmetric matrices,
- quadratic forms.

Sections 7.1–7.2 in "Linear Algebras and Its Applications" (pages 393–407).

