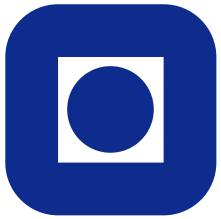

NTNU
Norwegian University of
Science and Technology

TMA4115 - Calculus 3

Lecture 25, April 17

Toke Meier Carlsen
Norwegian University of Science and Technology

Spring 2013

Review of last week's lecture

Last week we studied

- real matrices with complex eigenvalues,
- systems of first order differential equations.

NTNU
Norwegian University of
Science and Technology

Today's lecture

Today we shall introduce and study

- *the inner product*,
- the *length* of a vector,
- *orthogonality* and *orthogonal* sets in $\mathbb{R}^n$,
- *orthogonal* matrices.

NTNU
Norwegian University of
Science and Technology

The inner product

- If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$, then the *inner product* (or *dot product*) of $\mathbf{u}$ and $\mathbf{v}$ are defined by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$.
- If we regard $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

NTNU
Norwegian University of
Science and Technology

Properties of the inner product

Theorem 1

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

NTNU
Norwegian University of
Science and Technology

The length of a vector

- If $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is a vector in $\mathbb{R}^n$, then the *length* (or *norm*) of $\mathbf{v}$ is the nonnegative number $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$.
- A *unit vector* is a vector whose length is 1.
- If $\mathbf{v}$ is in $\mathbb{R}^n$ and c is a scalar, then $\|c\mathbf{v}\| = |c|\|v\|$.
- If $\mathbf{v}$ is a nonzero vector then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector.
- The process of creating $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ from $\mathbf{v}$ is called *normalizing* $\mathbf{v}$.
- If $\mathbf{u} = c\mathbf{v}$ where $c > 0$, then we say that $\mathbf{u}$ is *in the same direction* as $\mathbf{v}$.

NTNU
Norwegian University of
Science and Technology

Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Let us find a unit vector in the direction of $\mathbf{u}$ and a vector of length 2 in the direction of $-\mathbf{u}$.

Solution

$\|\mathbf{u}\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$, so

$\frac{1}{\sqrt{10}}\mathbf{u} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$ is a unit vector in the direction of $\mathbf{u}$, and $\frac{-2}{\sqrt{10}}\mathbf{u} = \frac{-2}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$ is a vector of length 2 in the direction of $-\mathbf{u}$.

Distance in $\mathbb{R}^n$

- If **u** and **v** are vectors in $\mathbb{R}^n$, then the *distance between* ***u*** *and* ***v***, written as dist(u, v), is the length of $\mathbf{u} - \mathbf{v}$.
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\| = \text{dist}(\mathbf{v}, \mathbf{u})$.

NTNU
Norwegian University of
Science and Technology

Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$

If **u** and **v** are nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$$

where θ is the angle between the two line segments from the origin to the points identified with **u** and **v**.

Orthogonal vectors and the Pythagorean theorem

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are *orthogonal* to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 2

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 2

$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} =$
$\|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\cdot\mathbf{v}$
so $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2$ if and only if $\mathbf{u}\cdot\mathbf{v}=0$.

NTNU
Norwegian University of
Science and Technology

Orthogonal complements

Let W be a subspace of $\mathbb{R}^n$.

- If a vector $\mathbf{z}$ is orthogonal to every vector in W, then $\mathbf{z}$ is said to be *orthogonal to* W.
- The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.
- If $W = \operatorname{Span} S$, then $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every $\mathbf{v}$ in S.
- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- $(W^\perp)^\perp = W$.

NTNU
Norwegian University of
Science and Technology

The orthogonal complements of Row(A) and Col(A)

Theorem 3

Let A be an $m \times n$ matrix. Then

1. $(\text{Row}(A))^{\perp} = \text{Nul}(A)$.
2. $(\text{Col}(A))^{\perp} = \text{Nul}(A^T)$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 3

Let $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$.

Then $A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$, so $\mathbf{x}$ is in $\text{Nul}(A)$ if and only if $\mathbf{x}$ is orthogonal to each row of A. Thus $(\text{Row}(A))^\perp = \text{Nul}(A)$.

It follows that $(\text{Col}(A))^\perp = (\text{Row}(A^T))^\perp = \text{Nul}(A^T)$.

NTNU
Norwegian University of
Science and Technology

Example

Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$. Let us determine $W^{\perp}$.

Solution

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $W = \text{Col}(A)$, so $W^{\perp} = \text{Nul}(A^T)$. To determine $\text{Nul}(A^T)$ we reduce A^T to its reduced echelon form.

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \end{bmatrix}$$

It follows that $W^{\perp} = \text{Nul}(A^T) = \text{Span}\left\{ \begin{bmatrix} 3 \\ -3/2 \\ 1 \end{bmatrix} \right\}$.

NTNU
Norwegian University of
Science and Technology

Orthogonal sets

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.

Theorem 4

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence a basis for Span S.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 4

Assume that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Then we have for each k that $0 = \mathbf{v}_k \cdot \mathbf{0} = \mathbf{v}_k \cdot (c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_k\mathbf{v}_k \cdot \mathbf{v}_k$. Since $\mathbf{v}_k \cdot \mathbf{v}_k = \|\mathbf{v}_k\| \neq 0$, it follows that $c_k = 0$. This shows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

NTNU
Norwegian University of
Science and Technology

Orthogonal bases

- An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.

Theorem 5

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{y}$ be in W. Then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 5

Let $\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ be the coordinate vector of $\mathbf{y}$ relative to $\mathcal{B}$. Then $\mathbf{y} = c_1\mathbf{v}_1 + \ldots c_p\mathbf{v}_p$, so $\mathbf{y} \cdot \mathbf{v}_j = (c_1\mathbf{v}_1 + \ldots c_p\mathbf{v}_p) \cdot \mathbf{v}_k = c_j\mathbf{v}_j \cdot \mathbf{v}_j$ for each j.

Since $\mathbf{v}_j \cdot \mathbf{v}_j = \|\mathbf{v}_j\|^2 \neq 0$, it follows that $c_j = \frac{\mathbf{y}\cdot\mathbf{v}_j}{\mathbf{v}_j\cdot\mathbf{v}_j}$.

NTNU
Norwegian University of
Science and Technology

Orthonormal sets

- An *orthonormal set* is an orthogonal set of unit vectors.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if and only if
$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
- An *orthonormal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal basis for W if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$.
- If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and $\mathbf{y}$ is in W, then the coordinates $c_1, \ldots, c_p$ of $\mathbf{y}$ relative to $\mathcal{B}$ is given by $c_j = \mathbf{y} \cdot \mathbf{v}_j$.

NTNU
Norwegian University of
Science and Technology

Matrices with orthonormal columns

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Proof of Theorem 6

Let $U = [\mathbf{v}_1 \dots \mathbf{v}_n]$. Then

$$U^T U = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \dots & \mathbf{v}_1 \cdot \mathbf{v}_n \\ \vdots & & \vdots \\ \mathbf{v}_n \cdot \mathbf{v}_1 & \dots & \mathbf{v}_n \cdot \mathbf{v}_n \end{bmatrix}$$

so $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal set if and only if $U^T U = I_n$.

Matrices with orthonormal columns

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
2. $\|U\mathbf{x}\| = \|\mathbf{x}\|$.
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

NTNU
Norwegian University of
Science and Technology

Proof of Theorem 7

Suppose $U^TU = I_n$ and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then
$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T(U\mathbf{y}) = \mathbf{x}^TU^TU\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.
It follows that $\|U\mathbf{x}\| = \sqrt{(U\mathbf{x}) \cdot (U\mathbf{x})} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \|\mathbf{x}\|$, and that
$(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

NTNU
Norwegian University of
Science and Technology

Tomorrow's lecture

Tomorrow we shall introduce and study

- *orthogonal projections*,
- the *Gram-Schmidt process*,
- *QR factorization*.

Sections 6.3–6.4 in "Linear Algebras and Its Applications" (pages 347-360).

NTNU
Norwegian University of
Science and Technology